Credit Spread Option Valuation under GARCH

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Working Paper 00-07
July 2000

ISSN : 1206-3304

Financial support by the Risk Management Chair is acknowledged. The author would like to thank his professors Peter Christoffersen, Georges Dionne, Geneviève Gauthier and Jean-Guy Simonato for their support and their comments.

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Abstract

This paper develops closed-form solutions for options on credit spreads with GARCH models. We extend the mean-reverting model proposed in Longstaff and Schwartz (1995) and we use the Heston and Nandi's (1999) GARCH specification rather than the traditional lognormal. Our model, being more flexible, captures better the empirical properties of observed credit spreads and contains Longstaff and Schwartz (1995) model as a special case. GARCH coefficients are estimated using spread levels for corporate bonds.

Keywords : Credit spread, options, GARCH models, mean-reversion.

Résumé

Cet article propose une formule fermée pour l'évaluation des options sur les écarts de crédit dans le cadre des modèles GARCH. On se base sur le modèle de retour à la moyenne proposé par Longstaff et Schwartz (1995) et on utilise le modèle GARCH proposé par Heston et Nandi (1999) au lieu du traditionnel modèle lognormal. Ce modèle étant plus flexible, s'ajuste mieux aux propriétés empiriques des données observées . Il contient le modèle de Longstaff et Schwartz (1995) comme cas particulier. Les coefficients du modèle GARCH sont estimés en utilisant les niveaux des écarts de crédits des obligations corporatives.

Mots clés : Écart de crédit, options, modèles GARCH, retour à la moyenne.

I Introduction

Until recently, credit risk was considered as legitimate to do business, just as another uncertainty factor which was unhedgeable. Currently, credit risk can be purchased, sold or restructured within portfolios in the same way as traditional financial products. This is made possible thanks to Credit Derivatives, which are the most important new types of financial products introduced during the last decade. These instruments offer investors an important new tool for dynamic hedging and managing long positions in credit risk exposures. From their characteristics, credit derivatives present several resemblances to the traditional options. One difficulty, however, comes from the fact that the determinant variable is the evolution of the credit spread instead of traditional interest rates or exchange rates. Thus, we need a model for credit spreads. Basing their justification on the observed empirical properties of credit spreads, Longstaff and Schwartz (1995) proposed a mean-reverting model for the logarithm of the credit spread. They showed that credit spreads were mean-reverting in logarithm but they assumed the change in logarithm to be well-approximated by the normal distribution.

This paper provides a more general framework for the volatility using GARCH models. It is well-known that financial data sets exhibit conditional heteroskedasticity. GARCH-type models are often used to model this phenomenon and show their ability to explain some irregularities (e.g. in equity returns) better than the traditional Geometric Brownian Motion. In the literature (see Bollerslev and al., 1992), there are many applications of ARCH models to interest rate data. All these applications focused on : 1) Term Spread, Engle and al. (1987) and Engle and al. (1990) estimated the relationship between long- and short-term interest rates; 2) Bond Yields Levels, Weiss (1984) estimated ARCH models on AAA corporate bond yields and found that ARCH effects were significantly evident. Although most studies involving interest rates used linear GARCH models, nonlinear dependencies could possibly exist in the conditional variance.

This paper estimates GARCH effects on the Credit Spread. We use the Heston and Nandi's (1999) GARCH specification and we keep the mean-reverting character showed by Longstaff and Schwartz (1995). The GARCH model, being more general for the volatility, fits observed credit spreads data better than the simple mean-reverting normal model. Also, Heston and Nandi (1999) showed that their conditional variance process converges weakly to Heston's (1993) Stochastic Volatility model, which means that our model has a mean-reverting square-root variance process as a continuous-time limit. Thus, our model contains continuous-time Longstaff and Schwartz (1995) model as a special case. Details on the convergence will be provided later.

The next section examines the mean-reversion character of credit spreads in our data set. Section III describes the GARCH process and presents credit spread options formulas. Section IV estimates GARCH coefficients with the maximum likelihood method using corporate bond spreads levels over treasuries, analyses some properties of the GARCH credit spread options and compares our results to those of Longstaff and Schwartz (1995). Calculation details are in the Appendix. Figures are presented at the end of the main text.

II Credit spread mean-reversion

We examine the spread between Moody's AAA and BAA 10-20 years bond indices and several U.S. Treasury bond yields (30 and 10 years maturity). We use daily observations over 1986-1992. Summary statistics for the spreads are presented in Table 1. We denote the logarithm of the credit spread by X_t. Figures 1 and 2 plot the time series of X_t for AAA and BAA indices over 30 years and 10 years U.S. TBond.

Table 1 : Summary statistics for AAA and BAA credit spreads and log-spreads over treasuries. September 1986 - December 1992

	30 years Tbond		10 years Tbond		Number of Observations
	Mean	**Std Dev**	**Mean**	**Std Dev**	
AAA	0.72239	0.21091	0.93317	0.25866	1654
Log AAA	- 4.9739	0.29977	- 4.7146	0.29016	1654
BAA	1.76570	0.34370	1.97640	0.34616	1654
Log BAA	- 4.0551	0.19144	- 3.9388	0.17187	1654

We also report Skewness and Kurtosis coefficients for log-spreads in Table 2. We can easily see that these coefficients are different from those of a normal distribution. This implies that we cannot assume a normal distribution for log-spreads. Instead, we propose a different process, such as GARCH, that takes into account this findings.

Table 2 : Skewness and Kurtosis coefficients for AAA and BAA log-spreads over treasuries. September 1986 - December 1992

	30 years Tbond		10 years Tbond		Number of observations
	Skewness	**Kurtosis**	**Skewness**	**Kurtosis**	
Log AAA	- 0.3060	2.9910	- 0.4901	3.5275	1654
Log BAA	0.1323	2.5354	0.1614	2.7811	1654

It is shown that both time series display mean-reversion. AAA credit spreads are apparently more mean-reverting than BAA. In order to formalize these observations and see how our data evolve over time, we regressed daily changes in the value of X_t on the value of X_t one day before :

$$DX_{t+1} = X_{t+1} - X_t = a + b\ X_t + e_t\ . \tag{1}$$

The regression results are reported in Table 3. The slope coefficient is significantly negative in all the regressions. The AAA slope coefficients are definitely higher than the correspondent BAA's, which implies that they are more mean-reverting.

Table 3 : Results from regressing daily changes in the logarithm of the credit spread of AAA and BAA over 30y and 10y US TBond

	a	***b***	**t-ratio *a* #**	**t-ratio *b* #**	***R*²***	***SE****
AAA Tb30y	-0.16803	-0.03365 {-3.1308}¤	-5.473 [5.11e-8]+	-5.461 [5.46e-8]	0.0177	0.07508
AAA Tb10y	-0.12701	-0.02691 {-2.9632}	- 4.818 [1.58e-8]	- 4.821 [1.56e-6]	0.0139	0.06582
BAA Tb30y	-0.05365	-0.01311 {-3.4497}	-3.642 [2.79e-4]	-3.613 [3.12e-4]	0.0078	0.02823
BAA Tb10y	-0.06134	-0.01551 {-3.0131}	-3.899 [1.05e-4]	-3.887 [1.06e-4]	0.0091	0.02788

* *SE* is the standard error of the regression and R^2 is the determination coefficient
# All the coefficients are significant at the 99% level
+ p-values are reported in brackets
¤ Dickey-Fuller test statistics are reported for Unit Root Test. The asymptotic critical values are -3.43 at 1% and -2.86 at 5%

Note also that the logarithm of the AAA credit spread is more volatile than that of the BAA's. The standard errors of regressions that used AAA bonds are higher than those of regressions that use BAA bonds. Longstaff and Schwartz (1995) found the same property with their data set. The values of R^2 are of the same order of magnitude than those reported by Longstaff and Schwartz (1995).

Given these empirical properties, one should assume a mean-reverting process for the logarithm of the credit spread. But unlike the Longstaff and Schwartz (1995) model and given the skewness and kurtosis analysis in Table 2, we propose a GARCH framework for the volatility of the logarithm of the credit spread. We use Heston and Nandi's (1999) GARCH(1,1) specification which is asymmetric. We also use their methodology to derive closed-form solutions for credit spread options. We assume that the riskless interest rate is constant and it is denoted by *r*.

III The model and the option valuation formula

We define X_t as the value of the logarithm of the credit spread at the end of period *t* and time periods are of length *D*. We assume that (X_t) follows the process given by :

$$\begin{aligned} X_{t+1} &= m + gX_t + l h_{t+1} + \sqrt{h_{t+1}}\, z_{t+1} \\ h_{t+1} &= b_0 + b_1 h_t + b_2\left(z_t - q\sqrt{h_t}\right)^2 \end{aligned} \tag{2}$$

or equivalently, to show the mean-reverting feature, by :

$$\begin{aligned} X_{t+1} - X_t &= m + (g-1)X_t + l h_{t+1} + \sqrt{h_{t+1}}\, z_{t+1} \\ h_{t+1} &= b_0 + b_1 h_t + b_2\left(z_t - q\sqrt{h_t}\right)^2 \end{aligned}$$

where h_t is the conditional variance of X_t known at time $t-1$ and $\left(z_t : t \in \{1,...,T\}\right)$ is a sequence of independent standard normal random variables. As pointed out by Heston and Nandi (1999), although this specification differs from the classic GARCH models, it is quite similar to the NGARCH model of Engle and Ng (1993). This model has the advantage that it provides closed-form solutions for the credit spread derivatives. When b_1 and b_2 are equal to zero, our model is equivalent to the Longstaff and Schwartz (1995) model observed at discrete intervals, with a risk-premium parameter l. This parameter was assumed to be equal to zero in Longstaff and Schwartz (1995) because their model was assumed to be risk-adjusted and the parameter m incorporated the market price of the risk-premium. We cannot use this assumption within our model because the volatility is not constant. As in Heston and Nandi's (1999) model, the parameter q controls for the skewness or the asymmetry of the distribution of the log-spreads. If $q > 0$, this implies that negative z_t raise the variance more than positive z_t. The covariance of the log-spread process and the variance process is given by :

$$Cov_{t-1}\left(X_t, h_{t+1}\right) = -2qb_2 h_t . \tag{3}$$

If the kurtosis parameter b_2 is positive, positive values for q result in negative correlation between the two processes.

As shown in Heston and Nandi (1999), the discrete time variance process h_t converges weakly to a variance process v_t that follows the square-root process of Feller (1951), Cox, Ingersoll and Ross (1985), and Heston (1993), when the time step length D tends to zero (see Foster and Nelson 1994). The log-spread process X_t will also have a continuous-time diffusion limit. Thus our two-processes model converges weakly (see *Convergence* in Appendix) to a mean-reverting square-root variance process (X,v) :

$$\begin{aligned} dX_t &= \left(h - d\, X_t + l\, v_t\right)dt + \sqrt{v_t}\, dZ_t \\ dv_t &= \left(w - k\, v_t\right)dt + s\sqrt{v_t}\, dZ_t \end{aligned} \tag{4}$$

where h is the long-run mean, d is the mean-reversion parameter, l is the risk-premium parameter, w, k and s are the square-root process parameters. By assuming that k, s and l are all zero, we get the constant volatility risk-adjusted Longstaff and Schwartz (1995) model. Thus our valuation model will contain the Longstaff and Schwartz (1995) valuation formula for constant risk-free rate as a special case. In order to value options, we must work under a risk-neutral probability measure. Let us rewrite Equation (2) in the form :

$$\begin{aligned} X_{t+1} &= m + gX_t + \sqrt{h_{t+1}}\, z_{t+1}^* \\ h_{t+1} &= b_0 + b_0 h_t + b_2\left(z_t^* - q^*\sqrt{h_t}\right)^2 \end{aligned} \tag{5}$$

where

$$z_t^* = z_t + l\sqrt{h_t}$$
$$q^* = q + l$$

Note that all we need is that $\left(z_t^* : t \in \{1,...,T\}\right)$ to be a sequence of risk-neutral independent random variables and that z_{t+1}^* to be a standard normal random variable conditional to the information available at time t (see *Derivation of the moment generating function* in Appendix) . This is obvious since h_{t+1} is known at time t and $\left(z_t : t \in \{1,...,T\}\right)$ is a sequence of independent standard normal random variables.

At this point, we can derive closed-form solutions for European options using the inversion of the characteristic function technique following Kendall and Stuart (1977). We first derive a formula for the characteristic function for the process given in Equation (2). Let $f(t,T,f)$ denote the moment generating function of $exp(X_T)$ under the historical probability measure conditional to the information available at time t :

$$f(t,T,f) = E_t\left(exp(f\ X_T)\right) \qquad (6)$$

where E_t denotes the time t conditional expectation operator under the historical probability measure. Motivated by the Heston and Nandi's (1999) asset price model, we calculate the moment generating functions for $exp(X_{t+1})$ and $exp(X_{t+2})$ to find that f takes a log-linear form and is a function of X_t and h_{t+1}. The general form for the moment generating function is given by :

$$f(t,T,f) = exp\left(g^{T-t} f\ X_t + A(t,T,f) + B(t,T,f)h_{t+1}\right) \qquad (7)$$

where

$$A(T,T,f) = B(T,T,f) = 0 \qquad (8)$$

and

$$A(t,T,f) = g^{T-t-1} m f + A(t+1,T,f) + b_0 B(t+1,T,f) - \frac{1}{2} ln\left(1 - 2b_2 B(t+1,T,f)\right) \qquad (9)$$

$$B(t,T,f)=g^{\,T-t-1}(q+l)f-\frac{1}{2}q^{2}+b_{1}B(t+1,T,f)$$
$$+\frac{1/2}{1-2b_{2}B(t+1,T,f)}\left(g^{\,T-t-1}f-q\right)^{2} \tag{10}$$

Given the terminal conditions, $A(t,T,f)$ and $B(t,T,f)$ can be calculated recursively. The Appendix derives the recursion formulas for these functions (see *Derivation of the moment generating function* in Appendix). Note that, although this formulation is given for the log-spread process under the historical probability measure, one can get the risk-neutral conditional moment generating function $f^{*}(t,T,f)$ by replacing q and l respectively by $q^{*}=q+l$ and $l^{*}=0$. The characteristic function of the process X_T under the historical probability measure conditional to time t is given by $f(t,T,if)$ where i is the complex number such that $i^{2}=-1$.

The value of a credit spread call with maturity T and strike price K is then given by :

$$C(t,T,X)=e^{-r(T-t)}E_{t}^{*}\left(\left(exp(X_{T})-K\right)^{+}\right) \tag{11}$$

where E_t^{*} denotes the time t conditional expectation operator under the risk-neutral probability measure and $(\bullet)^{+}$ denotes the positive part of a real number. We can compute the cumulative distribution function of the log-spread process by inverting its characteristic function. Indeed, Kendall and Stuart (1977) show that for a random variable Y :

$$P(Y\geq y)=\frac{1}{2}+\frac{1}{p}\int_{0}^{+\infty}Re\left(\frac{y^{-if}\,f(if)}{if}\right)\mathrm{d}f \tag{12}$$

where $Re(\bullet)$ denotes the real part of a complex number and f is the characteristic function. Using this result, Heston and Nandi (1999) provided a formula for the expected payoff. Evaluated under the risk-neutral probability measure, the call value is given by :

$$C(t,T,X)=e^{-r(T-t)}\left(f^{*}(1)P_{1}-KP_{2}\right) \tag{13}$$

where

$$P_1 = \frac{1}{2} + \frac{1}{p}\int_0^{+\infty} Re\left(\frac{K^{-if}\, f^*(if+1)}{if\; f^*(1)}\right)\mathrm{d}f$$

$$P_2 = \frac{1}{2} + \frac{1}{p}\int_0^{+\infty} Re\left(\frac{K^{-if}\, f^*(if)}{if}\right)\mathrm{d}f \tag{14}$$

This formula looks like the Longstaff and Schwartz (1995) formula. If we note that the $f^*(1)$ term represents, by definition, the risk-neutral forward (or expected) value of the spread

$$f^*(1) = f^*(t,T,1) = E_t^*\left(exp(X_T)\right), \tag{15}$$

and if we assume that the log-spread at time T is conditionally normal with mean u and variance h^2, we have :

$$E_t^*\left(exp(X_T)\right) = exp\left(u + \frac{h^2}{2}\right) \tag{16}$$

This term is present in the Longstaff and Schwartz (1995) formula and is equivalent to our $f^*(1)$ term in Equation (15). $P_2(\bullet)$ is the risk-neutral probability that the log-spread is greater than K at maturity. The call Delta ratio is given by (see *Derivation of delta* in Appendix) :

$$Delta = \frac{\partial C(t,T,X)}{\partial (e^X)} = e^{-r(T-t)} g^T f^*(1) P_1 \times e^{-X} \tag{17}$$

In the same way, the put value is given by (see *Credit spread Call-Put parity* in Appendix) :

$$P(t,T,X) = e^{-r(T-t)}\left(K(1-P_2) - f^*(1)(1-P_1)\right) \tag{18}$$

IV The empirical properties of credit spread options

This empirical section starts with the estimation of the GARCH coefficients using the data described earlier. It proceeds to analyze the implied conditional distribution. It then presents some properties of GARCH credit spread options and compares our model to Longstaff and Schwartz's (1995).

GARCH Estimation

For the estimation, we used daily data and set the time step length equal to 1 day. We used both AAA and BAA spreads over 10y and 30y US Tbond over September 1986 - December 1992. We estimated the coefficients using the maximum likelihood method. To illustrate the mean-reversion and the importance of the skewness parameter, we also estimated restricted models by setting first $g = 1$ and then $q = 0$. Restricted models and the unrestricted model are compared to each other using the log-likelihood ratio. Table 4 reports the estimation results. In all cases, the mean-reversion parameter, *g*, was significantly lower than 1. The restricted model $g = 1$ is strongly rejected in all cases. This reinforces the results reported in the regression analysis of the daily changes in the logarithm of the credit spread. The parameter *g* was also largely significantly different from 0. We also note that the *g*'s for the BAA bonds are higher than those of AAA. Thus, the AAA bonds are more mean-reverting than BAA, which was also reported in the regression analysis. For the skewness parameter, *q*, the restricted model $q = 0$ is easily rejected for AAA bonds and can not be rejected in the case of BAA over 30y Tbond. For BAA over 10y Tbond, the t-test rejects the hypothesis $q = 0$ while the log-likelihood ratio test does not. When the parameter q is significantly positive, this implies negative correlation between the log-spread and the volatility processes as shown in Equation (3). Log-spread processes over 10y Tbond have more skweness than those over 30y Tbond for both AAA and BAA bonds ($q = 16.196$ for AAA over 10y Tbond and $q = 14.765$ over 30y Tbond). The volatility process is stationary in all cases. Stationarity coefficients for AAA over 30y Tbond and AAA over 10y Tbond are quite similar. Thereafter, we used the estimates in the first line in Table 4 as our GARCH parameters. Figures 3 and 4 plot the volatility processes implied by the GARCH estimation for AAA bonds. For AAA bonds, the log-spread processes over both 30y and 10y Tbonds have quite similar stationarity coefficients $b_1 + b_2 q^2$ (0.911 $\cong$ 0.913) which measure the degree of the volatility mean-reversion (Heston and Nandi, 1999). Figures 5 and 6 plot the volatility processes implied by the GARCH estimation for BAA bonds. Comparing Figures 3 and 5 (or 4 and 6), we note that the volatility of AAA bonds is more mean-reverting than the BAA's. From Figures 5 and 6 and unlike AAA bonds, it is clear that the volatility of

Table 4 : Maximum log likelihood estimates of the GARCH model using daily data of the log-spread of AAA and BAA over 30y and 10y US Tbond

	m	g	l	b_0	b_1	b_2	q	$b_1+b_2q^2$ #	**Long-run volatility** ¤	**log L**	**L-ratio test g =1***	**L-ratio test q = 0***
AAA **Tb30y**	-0.153 (6.509) ** [1e-10] •	0.9697 (194.35) [0.000] {6.0675}+	4.477e-5 (7.09e-5) [1.000]	1.068e-4 (2.693) [7.2e-3]	0.827 (43.859) [3e-279]	3.812e-4 (10.685) [8.2^{e}-26]	14.765 (8.015) [2.1e-15]	0.911	0.0739	3602.95	46.10 [1.1e-11]	41.66 [1.1e-10]
AAA **Tb10y**	-0.145 (6.245) [5.4e-10]	0.9693 (186.81) [0.000] {5.9126}	4.442e-5 (6.87e-5) [1.000]	4.796e-5 (1.717) [8.6e-2]	0.839 (42.405) [2e-266]	2.836e-4 (10.543) [3.4^{e}-25]	16.196 (6.936) [5.8e-12]	0.913	0.0618	3882.28	51.18 [8.4e-13]	46.65 [8.5e-12]
BAA **Tb30y**	-0.038 (2.871) [4.2e-3]	0.9909 (301.34) [0.000] {2.7693}	1.217e-4 (8.14e-5) [1.000]	5.974e-5 (4.593) [4.7e-6]	0.771 (32.383) [1e-178]	1.086e-4 (11.441) [3.2^{e}-29]	3.768 (1.584) [0.113]	0.773	0.0272	5207.40	12.40 [4.3e-4]	0.7 [0.403]
BAA **Tb10y**	-0.055 (3.944) [8.4e-5]	0.9862 (274.27) [0.000] {3.8398}	9.924e-5 (6.96e-5) [1.000]	0.504e-5 (0.931) [0.352]	0.885 (117.36) [0.000]	0.729e-4 (13.316) [1.7^{e}-38]	9.270 (3.104) [1.9e-3]	0.892	0.0268	5234.42	14.6 [1.3e-4]	3.6 [0.0578]

* For the log-likelihood ratio test, the Chi-square critical values at 95% and 99% confidence levels are respectively 3.84 and 6.63

** t-ratios appear in parantheses. The t critical values at 95% and 99% confidence levels are respectively 1.645 and 2.326

• p-values are reported in brackets. For the log-likelihood ratio test, the p-values are those of the Chi-square distribution

\+ For g, we also report the t-ratio for testing $H_0 : g = 1$ vs. $H_1 : g < 1$

\# The Stationarity coefficient $b_1+b_2 q^2 < 1$ means that the volatility process is stationary

¤ Long-run volatility is the daily unconditional volatility given by $h_{unc} = (b_0+b_2) / (1-b_1-b_2 q^2)$

BAA over 10y Tbond is more mean-reverting than the volatility of BAA over 30y Tbond, which is indicated by a higher $b_1 + b_2 q^2$ (0.892 > 0.773).

Implied GARCH distribution :

Using Equation (12), we derived the implied conditional distribution of X_T. To see how this implied conditional distribution varies from the normal distribution, Figure 7 plots them for many variance ratios, that is the ratios between the initial variance and the unconditional variance denoted by h_{unc}. Higher ratio values imply fatter-tailed distributions. We can also expect that the conditional GARCH distribution has a positive skeweness, which seems to be the case for a ratio value of 1. Figure 8 plots the implied conditional distributions for a ratio value of 1 and for two different time horizons. The longer the time horizon, the higher the skeweness compared to the normal distribution.

Credit spread option properties :

We limit our discussion to call options when we analyze the properties of our model. In order to compare our discrete-time model to the continuous-time model of Longstaff and Schwartz (1995), we define their continuous-time parameters using the coefficients estimates such that the two stationary densities of processes X_t and h_t have the same first moments. The parameters in Equation (16) are defined as follows :

$$\begin{aligned} u &= X_0 g^T - \frac{m\left(1-g^T\right)}{ln(g)} \\ h^2 &= h_{unc}\left(1-g^{2T}\right) \end{aligned} \tag{19}$$

It is clear that when T takes high values, u and h^2 have finite limits because $g < 1$:

$$\begin{aligned} u_\infty &= \frac{-m}{ln(g)} \approx \frac{-m}{1-g} \\ h_\infty^2 &= h_{unc} \end{aligned} \tag{20}$$

The Longstaff and Schwartz (1995) call price (hereafter LS) is evaluated using the conditionally normal distribution for X_T with mean u and variance h^2. Figure 9 plots the call value as a function of the underlying credit spread using the GARCH parameters reported in Table 4 (line 1) and their risk-neutral counterparts, with a strike $K = 1$, $r = 0.1$ and for different variance ratios. The LS call price and the intrinsic value are also represented. An important property of GARCH credit spread calls, already noticed by Longstaff and Schwartz (1995) within their model, is that their value can be less than the intrinsic value, which is impossible with the Black and Scholes model. This is due to the mean-reversion character of the credit log-spreads. Intuitively, in-the-money calls are less likely to remain in the money over time, because the credit spread tends to decline towards its long-run mean. For variance ratios less than 1, the GARCH credit spread call prices are less than the LS price, while variance ratios greater than 1 give higher GARCH prices. As the underlying credit spread increases, the difference between call prices becomes small. Figure 10 plots the difference between LS and GARCH credit spread call prices for different variance ratios. Note that the higher the variance ratio, the greater the difference, which reinforces the results of Figure 9. The difference is , however, more important around the at-the-money calls. Figures 11 and 12 plot credit spread call prices for different maturities (10 days and 1 year). Figure 11 shows again that the call prices pass below the intrinsic value even when the call is only slightly in-the-money. Figure 12 gives another important property of GARCH credit spread calls. They can be concave functions of the underlying credit spread. Because of mean-reversion, the dynamics of the credit spread do not satisfy the first-degree homogeneity property necessary for options to be convex functions (Merton, 1973). In turn, this means that the delta of a GARCH credit spread call, given in Equation (17), could be a decreasing function of the underlying credit spread. The delta of a GARCH credit spread call decreases to zero as the time to maturity increases. Although our model is GARCH mean-reverting, these properties were somewhat expected since Longstaff and Schwartz (1995) have found that their continuous-time model exhibit such characteristics.

V Conclusion

We have proposed a GARCH mean-reverting model for credit log-spreads as an extension of the Longstaff and Schwartz (1995) model which uses a constant volatility. We used the Heston and Nandi's (1999) GARCH specification to allow the variance process to depend on the past levels of the credit spread. The GARCH was estimated using the maximum likelihood method and the important coefficients, especially the mean-reversion and the skeweness parameters, were found to be significant. Our model is then more flexible and captures the empirical properties of credit spreads in a better way than the traditional lognormal model. We also derived closed-form solutions for European options on credit spreads. Call prices exhibit the same unusual properties found by Longstaff and Schwartz (1995). The call value can be less than its intrinsic value and can be a concave function of the underlying credit spread. Comparing our model to Longstaff and Schwartz (1995) model, we have found that the difference between them is more important for at-the-money calls.

Although the closed-form solutions derived here are only for simple calls and puts on credit spreads within a GARCH framework, valuation expressions for other credit spreads European exotic derivatives, such as barrier options, could be derived in the same way.

Figure 1 : Moody's AAA and BAA log-spread over 30y US TBond

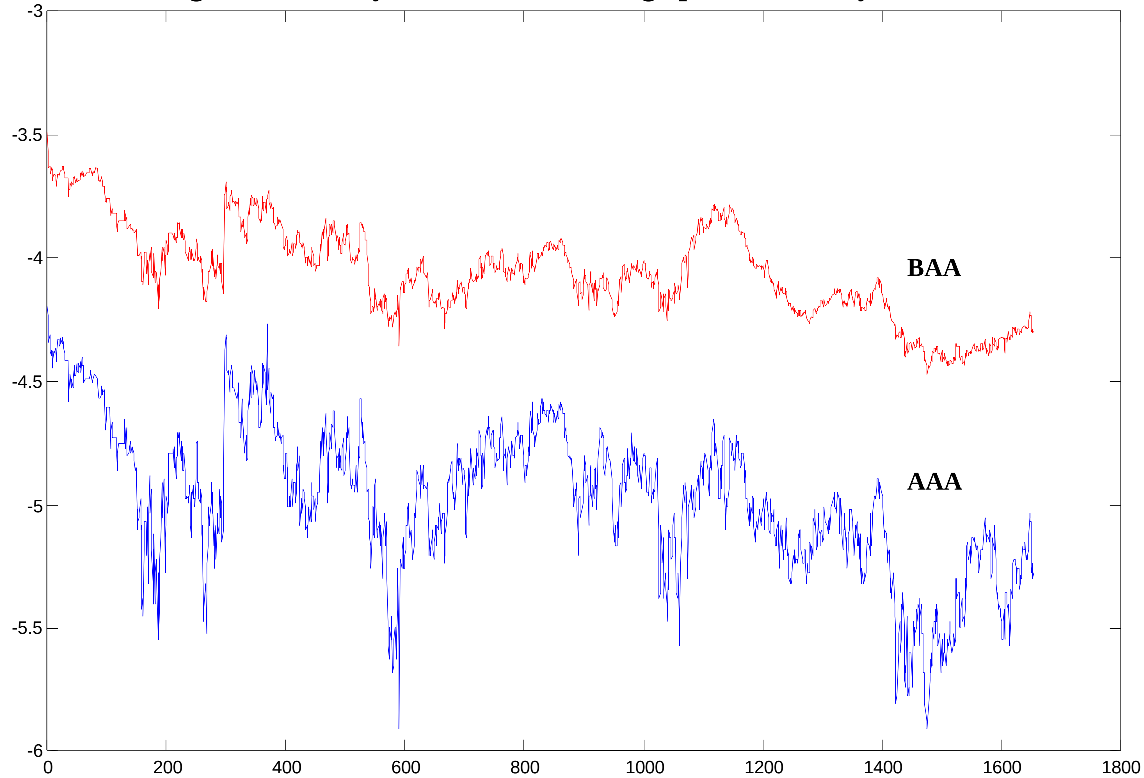


Figure 2 : Moody's AAA and BAA log-spread over 10y US TBond

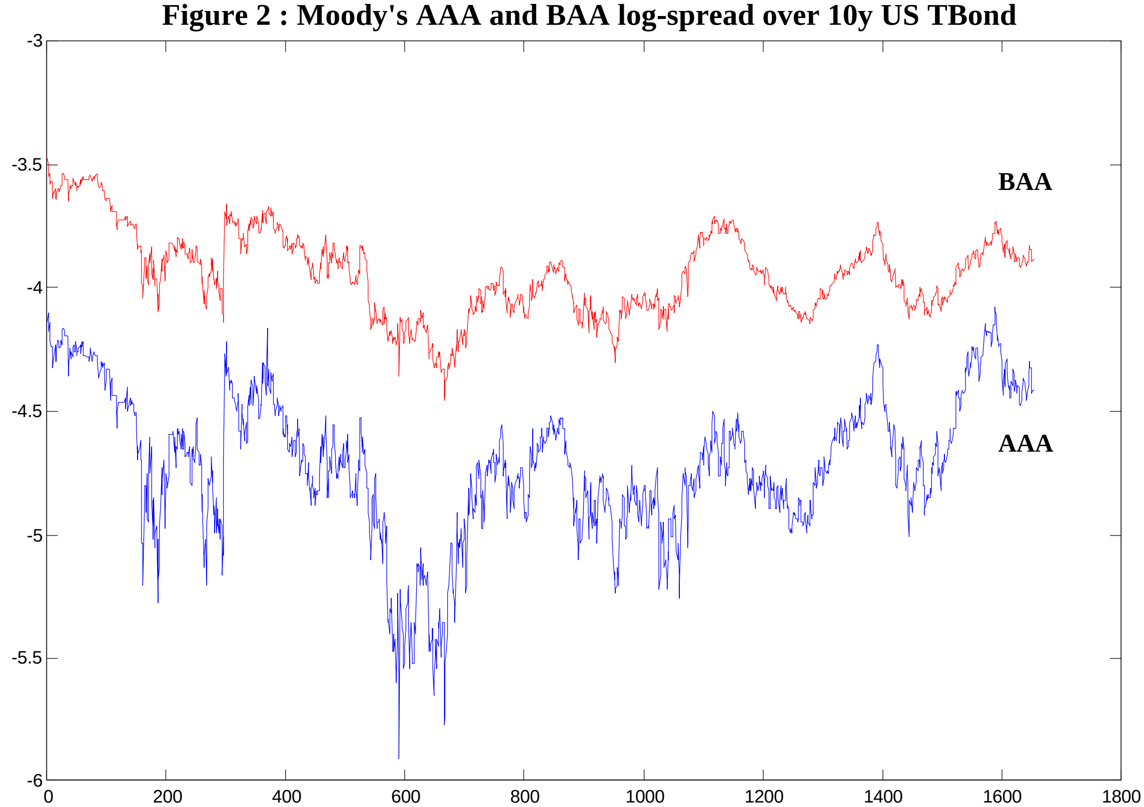


Figure 3 : Moody's AAA log-spread over 30y US Tbond - Volatilities

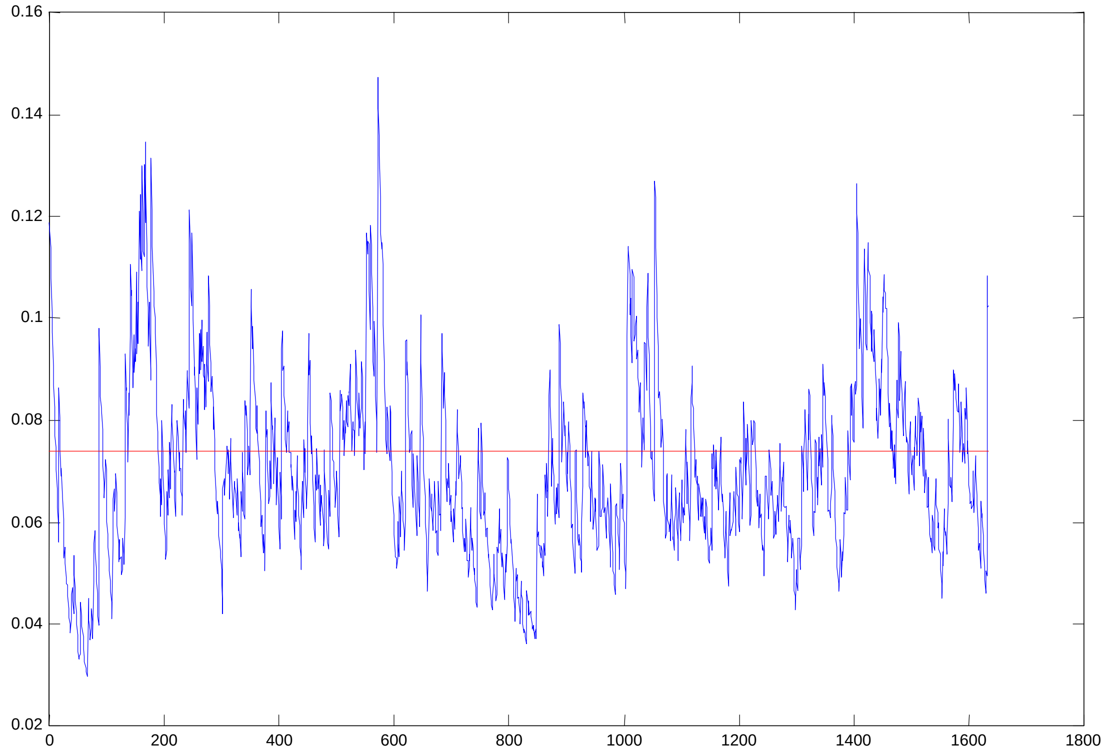


Figure 4 : Moody's AAA log-spread over 10y US Tbond - Volatilities

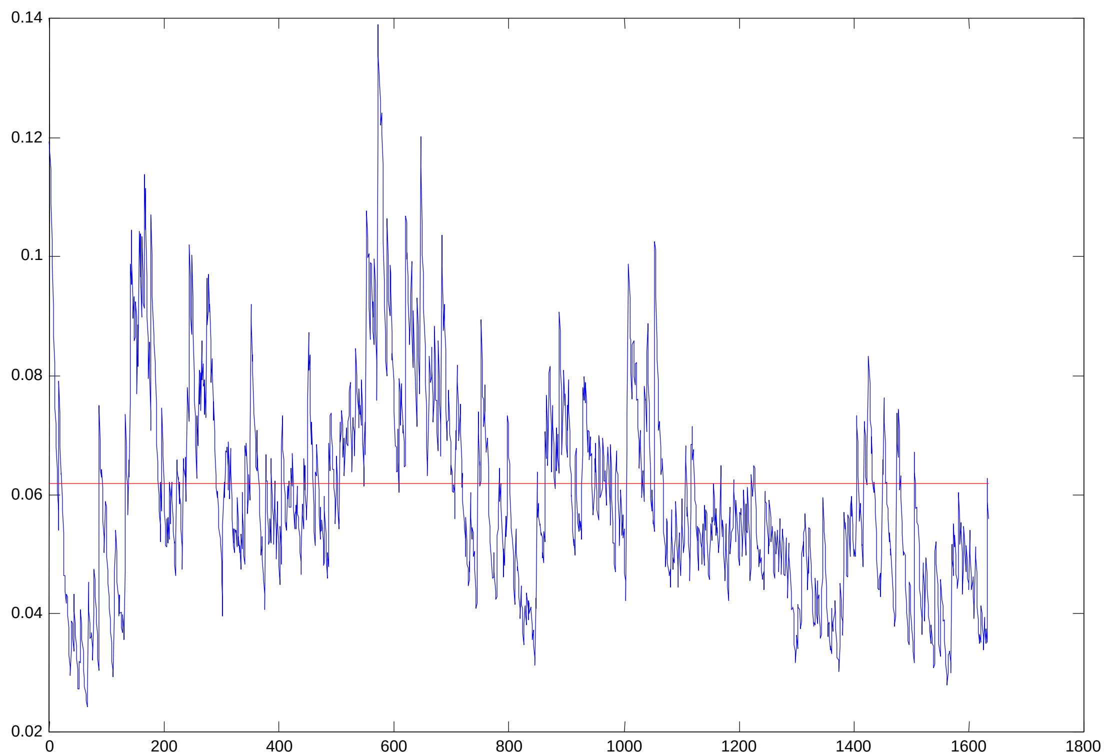


Figure 5 : Moody's BAA log-spread over 30y US Tbond - Volatilities

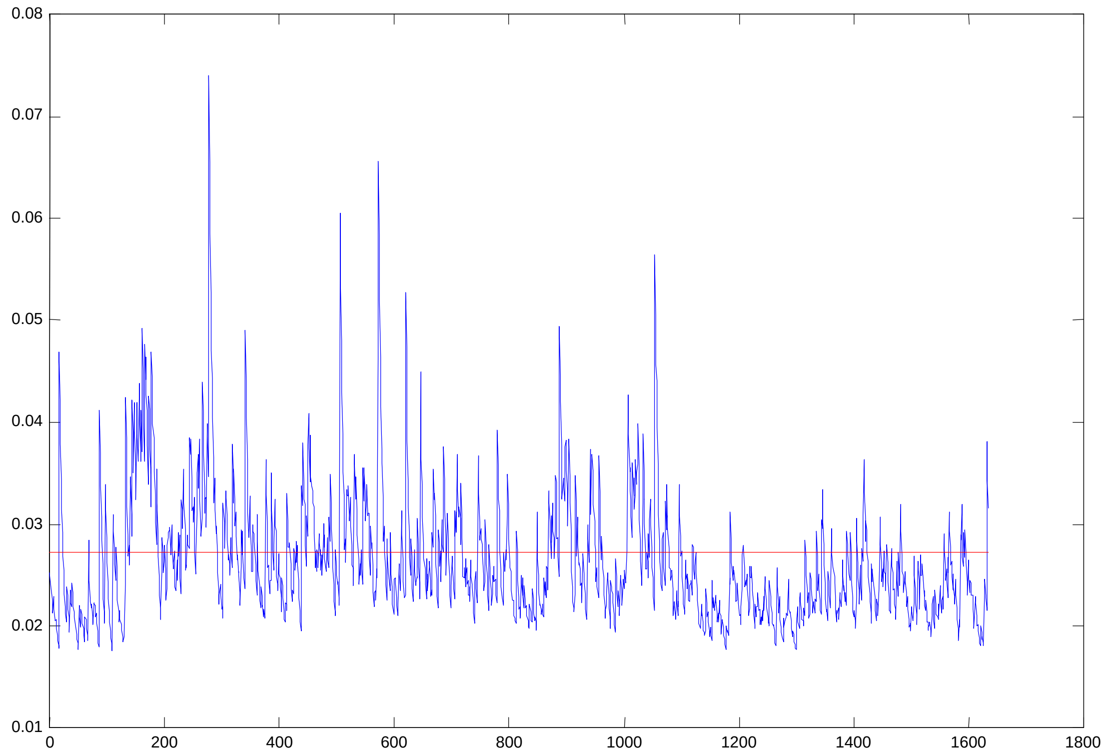


Figure 6 : Moody's BAA log-spread over 10y US Tbond - Volatilities

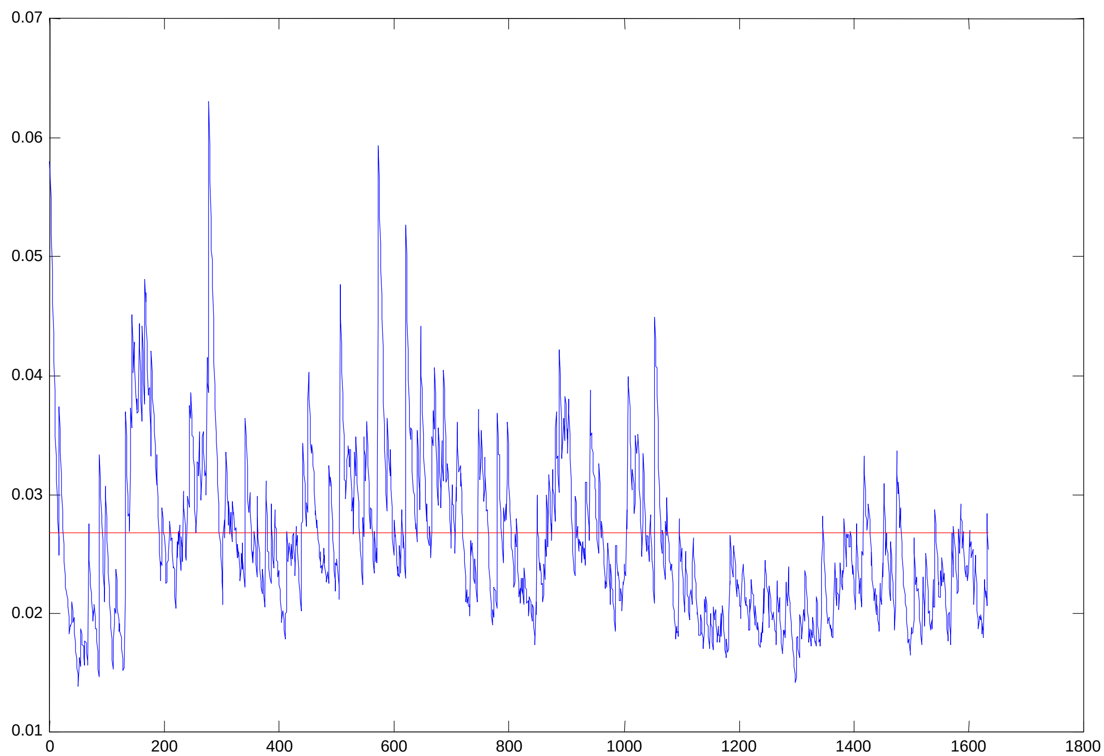


Figure 7 : Implied conditional log-spread distribution for different variance ratios

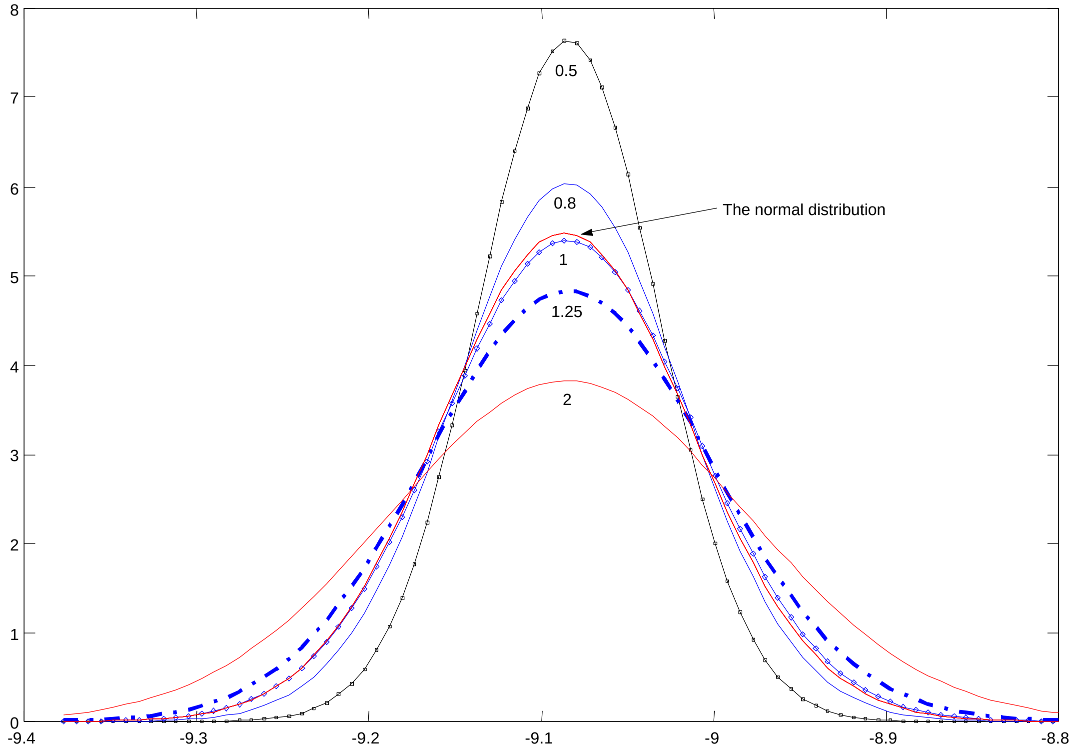


Figure 8 : Implied conditional log-spread distribution for different maturities

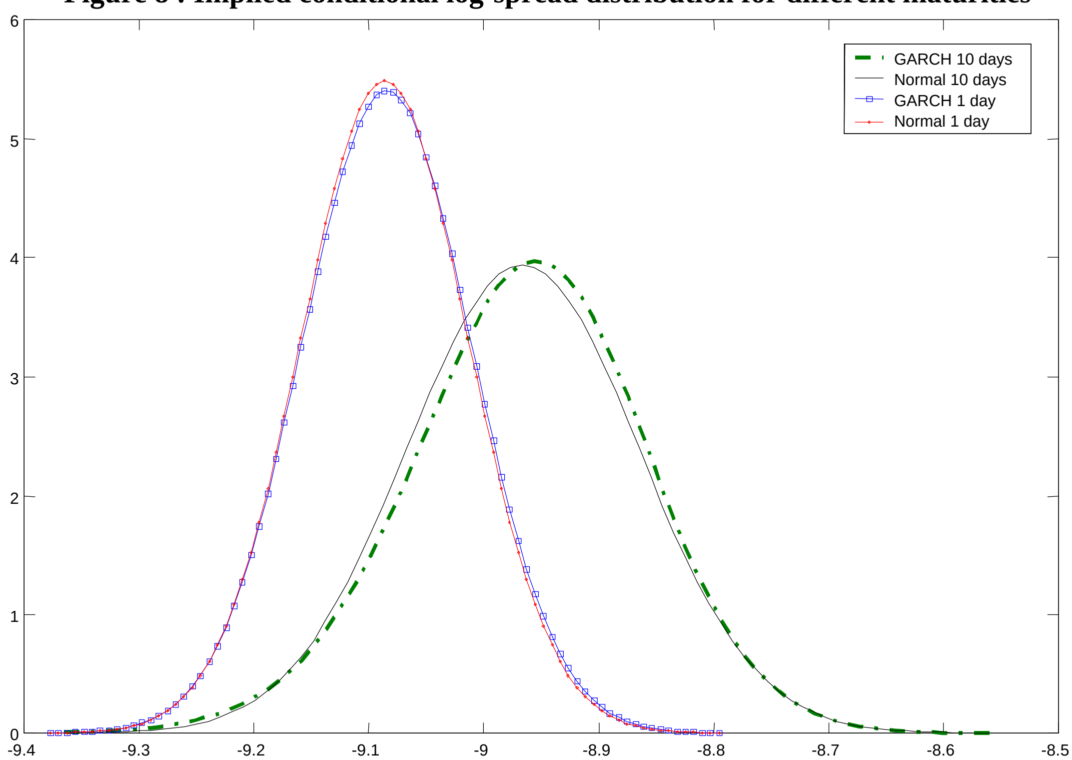


Figure 9 : Credit spread call prices for different variance ratios

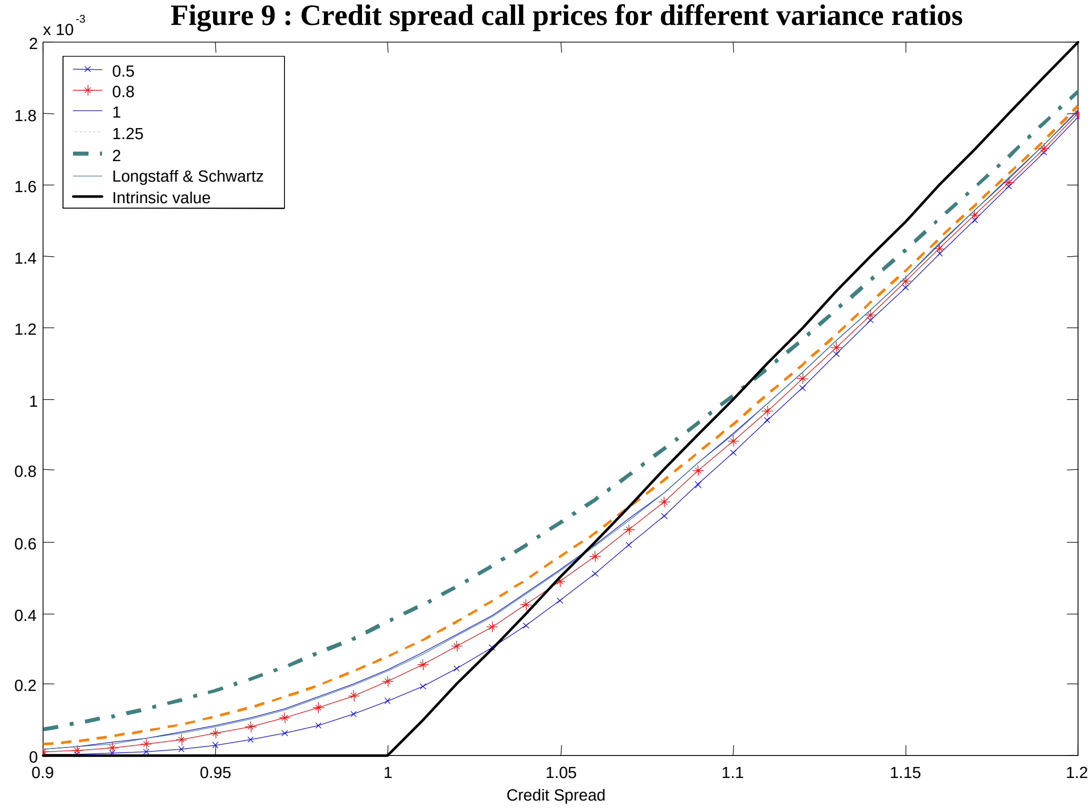


Figure 10 : Credit spread call price difference between GARCH and Longstaff and Schwartz model for different variance ratios

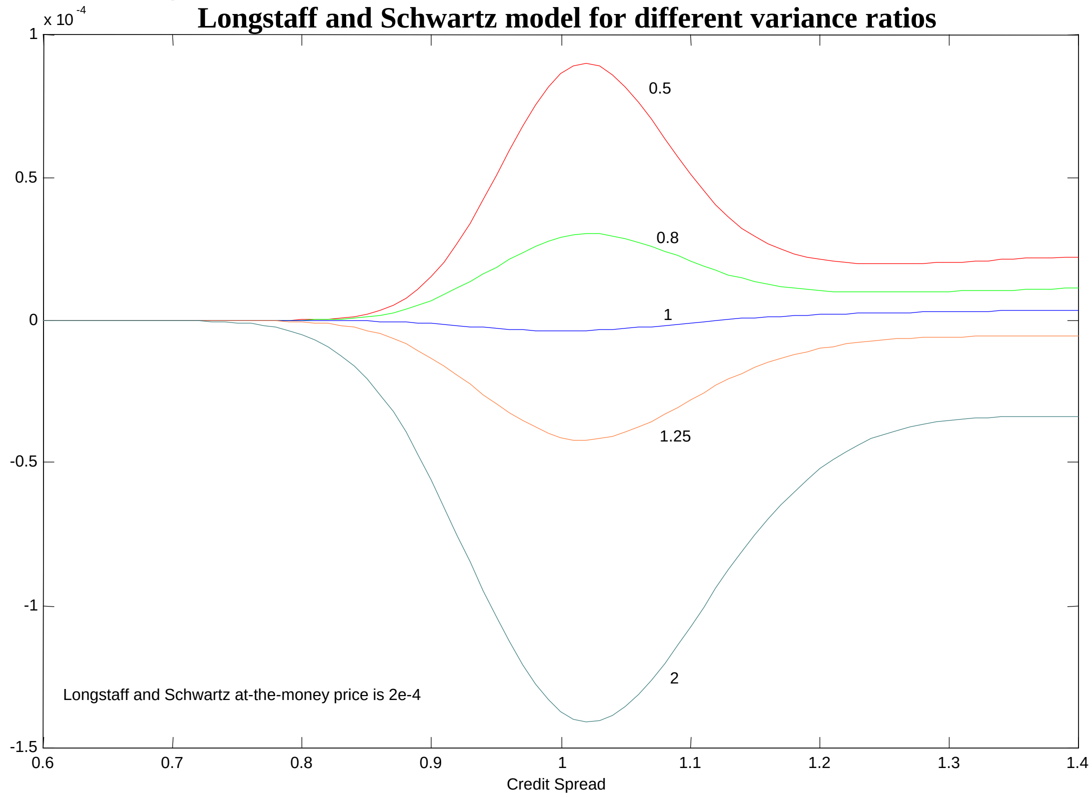


Figure 11 : Credit spread call prices for a different maturity

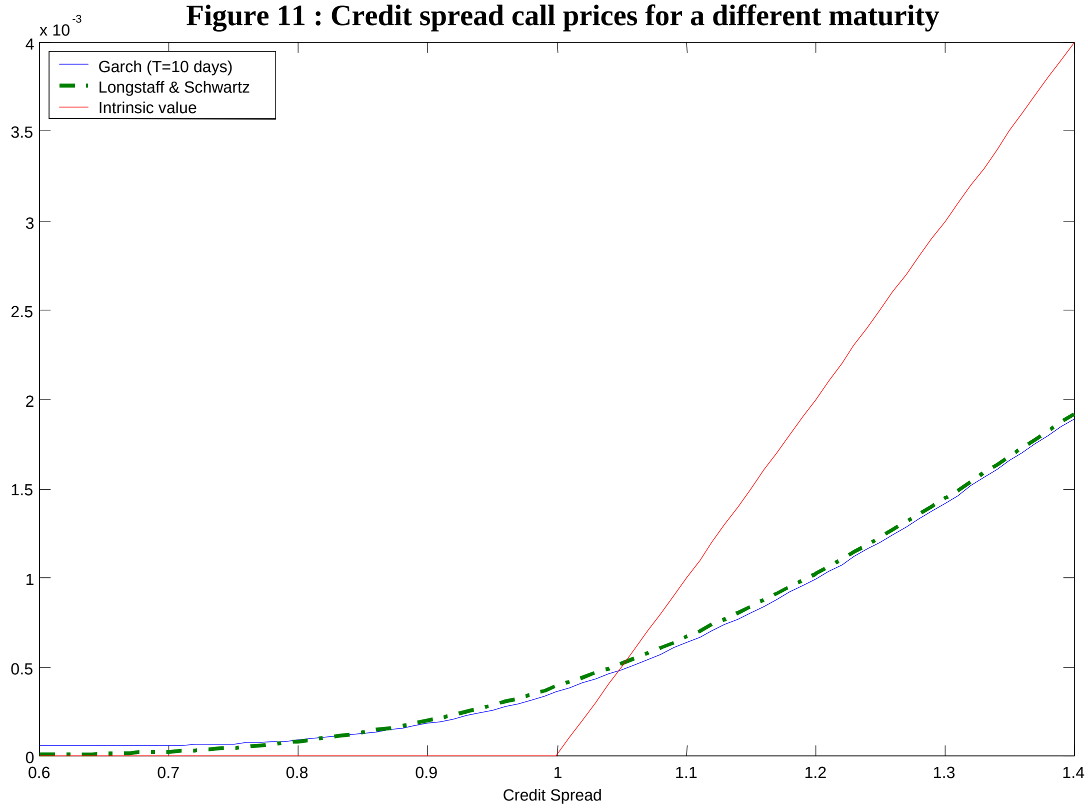
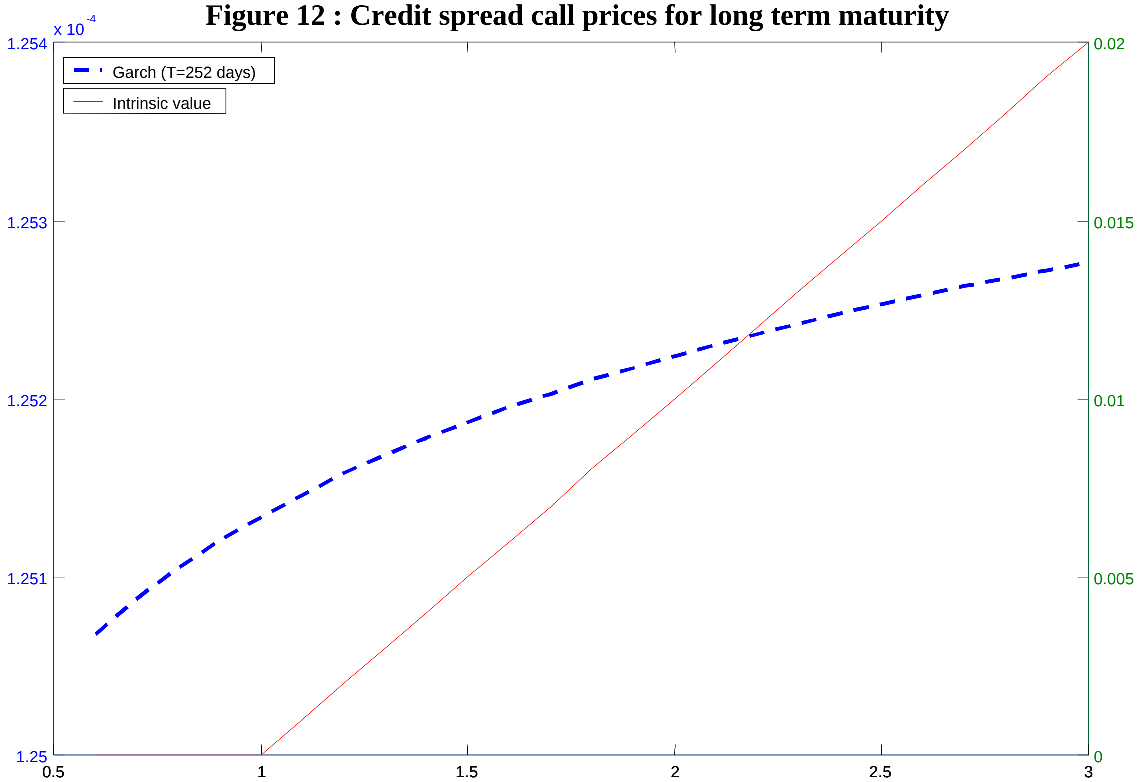


Figure 12 : Credit spread call prices for long term maturity



Appendix

Convergence :

Our model converges weakly to a continuous-time model. First, given the methodology in Heston and Nandi (1999), when the time step length D tends to zero, the variance process in our GARCH model converges weakly to a square-root process of Feller (1951), Cox, Ingersoll and Ross (1985) and Heston (1993). Define v as the limit process of variance h. Now, we have to show that the log-spread process X also converges weakly to a continuous-time process. The dynamics of X can be written as :

$$X_{t+1} - X_t = (h - dX_t + l v_{t+1})D + \sqrt{D}\sqrt{v_{t+1}} z_{t+1}$$

where $v_{t+1} = \frac{h_{t+1}}{D}$ and $d = 1 - g$. In the same way as in Heston and Nandi (1999) who worked with the special case $d = 0$ (no mean-reversion), this process converges weakly to :

$$\begin{aligned} dX_t &= (h - d\, X_t + l v_t)dt + \sqrt{v_t}\, dZ_t \\ dv_t &= (w - k\, v_t)dt + s\, \sqrt{v_t}\, dZ_t \end{aligned}$$

where (Z_t) is a Wiener process, and w, s and k are defined as in Heston and Nandi (1999).

Derivation of the moment generating function :

Recall that the moment generating function of the spread with a time horizon T conditional to the time t information is given by :

$$f(t,T,f) = E_t(exp(f\, X_T)).$$

First, we calculate this function for $T = t+1$ in order to get the log-linear form. Without any loss of generality, we assume that the time step length D is equal to 1. We then have :

$$\begin{aligned} f(t,t+1,f) &= E_t(exp(f\, X_{t+1})) \\ &= E_t\left(exp\left(fm + f g\, X_t + f l\, h_{t+1} + f\sqrt{h_{t+1}} z_{t+1}\right)\right) \\ &= exp(fm + f g\, X_t + f l\, h_{t+1}) E_t\left(exp\left(f\sqrt{h_{t+1}} z_{t+1}\right)\right) \end{aligned}$$

We obtain a log-linear form for $f(t,t+1,f)$ by noting that z_{t+1} is conditionally normally distributed and that h_{t+1} is known at time t. Hence we can write :

$$f(t,t+1,f) = exp\left[fm + f g\ X_t + \left(l f + \frac{1}{2} f^2 \right) h_{t+1}\right].$$

In the same way, one can calculate $f(t,t+2,f)$ and find that we still obtain a log-linear form for the generating function. Let us assume that at time $t+1$, we have :

$$f(t+1,T,f) = exp\left(f g^{T-t-1} X_{t+1} + A(t+1,T,f) + B(t+1,T,f)\, h_{t+2}\right)$$

where A and B are deterministic functions. At time t , the generating function can be written, using the iterated expectations law :

$$\begin{aligned} f(t,T,f) &= E_t\left(exp(f\ X_T)\right) \\ &= E_t\left(E_{t+1}\left(exp(f\ X_T)\right)\right) \\ &= E_t\left(f(t+1,T,f)\right) \end{aligned}$$

Then we have by definition of $f(t+1,T,f)$:

$$f(t,T,f) = E_t\left\{exp\left(f g^{T-t-1} X_{t+1} + A(t+1,T,f) + B(t+1,T,f)\, h_{t+2}\right)\right\}.$$

Replacing X_{t+1} and h_{t+2} by their expressions as functions of X_t and h_{t+1} (see Equation (2)), we get :

$$f(t,T,f) = E_t\left\{\begin{aligned} & exp\left(f g^{T\text{-}t} X_t + fmg^{T-t-1} + A(t+1,T,f) + b_0 B(t+1,T,f)\right) \\ & \times exp\left(b_1 B(t+1,T,f) h_{t+1} + f l g^{T-t-1} h_{t+1}\right) \\ & \times exp\left(b_2 B(t+1,T,f)\left(z_{t+1} - q\sqrt{h_{t+1}}\right)^2 + fg^{T-t-1}\sqrt{h_{t+1}}\, z_{t+1}\right) \end{aligned}\right\}$$

The two first terms in the expectation operator are known at time t, hence the only term that we need to compute is the last one. This is where the Heston and Nandi's (1999) GARCH specification takes the advantage over other GARCH models. The last term can be computed as a function of h_{t+1} using the fact that, for a standard normal variable z and constants a and b, we have :

$$E\left(exp\left(a(z+b)^2\right)\right) = exp\left(-\frac{1}{2} ln(1-2a)\right) \times exp\left(\frac{ab^2}{1-2a}\right)$$

Under the risk-neutral probability measure, we only need that z^*_{t+1} must be a standard normal variable conditional to time t and this is the case since h_{t+1} is known and is constant conditional to time t. Thus, the last term can be developed and rearranged to obtain a "perfect" square of z_{t+1} and h_{t+1}, added to a remaining term that depends on h_{t+1} and the GARCH parameters. Rearranging the terms in a log-linear form, we get Equations (9) and (10) in the main text.

Derivation of delta :

Note that the delta of a credit spread call is, by definition, equal to :

$$Delta = \frac{\partial C(t,T,X)}{\partial (e^X)} = \frac{\partial C(t,T,X)}{\partial X} e^{-X} .$$

Using Equation (13) that gives the call valuation expression, we can write :

$$e^{r(T-t)} \frac{\partial C(t,T,X)}{\partial X} = \frac{\partial f(1) P_1}{\partial X} - K \frac{\partial P_2}{\partial X} .$$

Note that

$$\frac{\partial f(t,T,f)}{\partial X} = f g^{T-t} f(t,T,f) .$$

We then have using Equation (14) :

$$\frac{\partial P_2}{\partial X} = \frac{g^{T-t}}{p} \int_0^{+\infty} Re\left(K^{-if} f(if)\right) \mathrm{d}f \qquad \text{(A1)}$$

and

$$\frac{\partial f(1) P_1}{\partial X} = g^{T-t} f(1) P_1 + \frac{g^{T-t}}{p} \int_0^{+\infty} Re\left(K^{-if} f(if+1)\right) \mathrm{d}f . \qquad \text{(A2)}$$

In order to obtain Equation (17), we use a change of variable ($j = f + i$) in Equation (A1) to see that :

$$K \frac{\partial P_2}{\partial X} - \frac{g^{T-t}}{p} \int_0^{+\infty} Re\left(K^{-if} f(if+1)\right) \mathrm{d}f = 0$$

The only remaining term, the first on the right-hand side of Equation (A2), multiplied by the discount factor and by $exp(-X)$ gives the delta expression as in Equation (17).

Credit spread Call-Put parity :

As for Black and Scholes model, we proceed by noting that the payoff of a portfolio that is long in a call and short in a put is given by :

$$\left(e^{X}-K\right)^{+}-\left(K-e^{X}\right)^{+}=e^{X}-K$$

Thus, the time *t* portfolio's value is simply the discounted forward value, that is :

$$e^{-r(T-t)}E_t^*\left(e^{X}-K\right)=e^{-r(T-t)}\left(E_t^*\left(e^{X}\right)-K\right)$$

which is equal to :

$$e^{-r(T-t)}\left(f^{*}(1)-K\right).$$

This means that :

$$C(t,T,X)-P(t,T,X)=e^{-r(T-t)}\left(f^{*}(1)-K\right).$$

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