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Working Paper 00-10
October 2000

ISSN : 1206-3304

The authors thank the Ministère de l'Éducation du Québec, *FCAR (Québec), RCM$_2$ (Canada), the FFSA (France) and the FNRS (Belgium) for financial support and C. Gollier, P. Gonzalez, B. Salanié and K. Viscusi for their comments on an earlier version.*

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Abstract

Recently, Caballé and Pomansky (1996) proposed a formal definition of mixed risk aversion and characterized stochastic dominance in presence of such utility functions. However they did not study comparative mixed risk aversion. In this note we give a sufficient condition for analytic comparative mixed risk aversion.

Keywords: Mixed risk aversion, comparative mixed risk aversion, self-protection, willingness to pay, proper risk behavior.

JEL Classification: D80.

Résumé

Récemment, Caballé et Pomansky (1996) ont proposé une définition formelle de l'aversion au risque mélangée et ont caractérisé la dominance stochastique pour cette classe de fonctions d'utilité. Par contre, ils n'ont pas étudié les comparaisons d'aversion au risque mélangée entre les individus. Dans cette note, nous proposons une condition suffisante pour permettre ce genre de comparaison.

Mots clés : Aversion au risque mélangée, comparaison de l'aversion au risque mélangée, autoprotection, volonté à payer, comportement cohérent face au risque.

Classification JEL : D80.

1 Introduction

For many economic applications under risk and uncertainty, a simple concave transformation of a von Newmann-Morgenstern utility function (or an Arrow-Pratt increase in risk aversion) does not yield intuitive changes in decision variables or in lottery choices by risk averse individuals.

In 1987, Pratt and Zeckhauser introduced the concept of Proper Risk Aversion in order to make prediction of lottery choices in presence of an independent, undesirable lottery or of an independent background risk. Their concept is preserved in the class of utility functions that are completely monotone or whose derivatives alternate in sign, with positive odd derivatives and negative even derivatives. They showed that mixtures of exponential utilities are proper. Brockett and Golden [1987] developed a parallel characterization of such functions and Hammond [1974] proposed a first application using a mixture (discrete) of exponential functions.

Recently, Caballé and Pomansky [1996] extended the analysis by characterizing stochastic dominance in presence of mixed risk aversion. They applied their model to the standard portfolio choice and provided a new set of sufficient conditions to obtain that a mixed risk averse individual will decrease his risky position when the risk increases. However, they did not study comparative mixed risk aversion. The objective of this paper is to propose such a transformation for mixtures of exponential functions or for utility functions exhibiting mixed risk aversion.

In Section 2 we introduce the concept of mixed risk aversion and in Section 3 we define that of "more mixed risk aversion". We present a transformation theorem that sets a sufficient condition to compare mixed risk averse utility functions.

By definition, individual v is more mixed risk averse than individual u if he is more risk averse, more prudent, more temperent ... or if the absolute ratio of the n^{th+1} derivative of v over the n^{th} is higher than the corresponding ratio of individual u for all n greater than one. As applications, Dachraoui et al. [2000] show how the concept of more mixed risk aversion is useful to make comparisons of risky decisions between risk averse individuals or to show that a more mixed risk averse individual will have a Proper Risk Behavior with respect to self-protection and willingness to pay. These applications are discussed in the concluding comments.

2 Mixed Risk Aversion

Caballé and Pomansky [1996] generalized the Arrow-Pratt index of absolute risk aversion to higher order. They defined the n^{th} order index of absolute risk aversion as

$$A_n^u(w) = -\frac{u^{(n+1)}(w)}{u^{(n)}(w)}, \text{ for } n \geq 1,$$

and defined mixed risk aversion as:

Definition 1 (*Caballé and Pomansky,* 1996) *A real-valued continuous utility function u defined on $[0,\infty)$ exhibits mixed risk aversion if and only if it has a completely monotone first derivative on $(0,\infty)$ $\left((-1)^{n+1}u^{(n)}(w) \geq 0\right.$, for $\left.n \geq 1\right)$, and $u(0) = 0$.*

As pointed out by Pratt and Zeckhauser [1987], most known utility functions that are commonly used in economics and finance such as the logarithmic and the power functions are in the class of completely monotone functions. (See Brocket and Golden, 1987 for other examples). These functions have the property of being characterized by the measure describing the mixture of exponential functions:

For any mixed risk averse utility function $u(w)$, there exists a distribution function F_u satisfying

$$\int_1^\infty \frac{dF_u(t)}{t} < \infty$$

with

$$u(w) = \int_0^\infty \frac{1-e^{-tw}}{t} dF_u(t).$$

3 Comparative mixed risk aversion

Let us consider two risk averse agents u and v. Following Arrow and Pratt, it has been established that comparative risk aversion can be reduced to applying a simple concave transformation of utility functions: v is more risk averse than u if and only if $v = k(u)$ with $k'' < 0$. However, this type of comparison is not sufficient for problems that imply at least the first two moments of the distribution such as self-protection choice or the comparison of willingness to pay. It is also not sufficient to compare levels of actions between different risk averse agents in standard principal-agent problems. As we will see it is also not sufficient to make comparison between mixed risk averse functions. We introduce the next definition:

Definition 2 *Let u and v be two mixed risk averse utility functions. We say that v is more mixed risk averse than u if and only if $A_n^u(w) \leq A_n^v(w)$, for all n and w.*

We propose the next transformation theorem.

Theorem 3 *Let u and v be two mixed risk averse utility functions described respectively by distribution functions F_u and F_v. If $\dfrac{dF_u(.)}{dF_v(.)}$ is decreasing over $(0, \infty)$, then v is more mixed risk averse than u.*

Before presenting the proof, let us consider a concrete example:

$$\begin{aligned} u(w) &= -p_1 e^{-a_1 w} - p_2 e^{-a_2 w} - ... - p_n e^{-a_n w}, \\ v(w) &= -q_1 e^{-a_1 w} - q_2 e^{-a_2 w} - ... - q_n e^{-a_n w} \end{aligned}$$

with p_i, q_i, and a_i as positive parameters for $i = 1, ...n$ and $a_1 < a_2 < ... < a_n$. If $\frac{p_1}{q_1} \geq ... \geq \frac{p_n}{q_n}$, then from Theorem 1 we know that v is more risk averse, more prudent, more temperent than u, and more generally

$$A_n^u(.) \leq A_n^v(.), \text{ for all } n \geq 1.$$

The intuition behind Theorem 3 is quite simple. Observe that u and v are mixtures of $CARA$ utility functions. Consider for the sake of illustration the case where there are only two positive a_i (a_1 and a_2) in the example above. By transforming p_1 into q_1 lower than p_1, less weight is put upon the less risk averse $CARA$ component of the u function (since $a_1 < a_2$). Of course lowering p_1 also implies that q_2 exceeds p_2 so that simultaneously more weight is placed upon the more risk averse component of u. As result v is surely more risk averse than u and the theorem shows that this property automatically extends to all ratios of successive derivatives of each utility function.

Proof.

We need to show that

$$\frac{\int_0^\infty t^{n+1}e^{-wt}dF_u\left(t\right)}{\int_0^\infty t^{n}e^{-wt}dF_u\left(t\right)} \leq \frac{\int_0^\infty t^{n+1}e^{-wt}dF_v\left(t\right)}{\int_0^\infty t^{n}e^{-wt}dF_v\left(t\right)},\ \text{for } n \geq 0.$$

The latter is equivalent to

$$\int_0^\infty td\widetilde{F}_u^{n,w}\left(t\right) \leq \int_0^\infty td\widetilde{F}_v^{n,w}\left(t\right), \tag{1}$$

where

$$d\widetilde{F}_i^{n,w}\left(t\right) = \frac{t^n e^{-wt}dF_i\left(t\right)}{\int_0^\infty s^n e^{-ws}dF_i\left(s\right)},\ \text{for } i = u,\ v.$$

Inequality (1) is equivalent to[1]

$$\int_0^\infty [1 - \widetilde{F}_u^{n,w}\left(t\right)]dt \leq \int_0^\infty [1 - \widetilde{F}_v^{n,w}\left(t\right)]dt,$$

or

$$\int_0^\infty [\widetilde{F}_v^{n,w}\left(t\right) - \widetilde{F}_u^{n,w}\left(t\right)]dt \leq 0. \tag{2}$$

A sufficient condition to have (2) is to show that

$$\widetilde{F}_v^{n,w}\left(t\right) \leq \widetilde{F}_u^{n,w}\left(t\right),\ \text{for all } t \text{ and all } n. \tag{3}$$

For a complete monotone function the last inequality simplifies to

$$\frac{\int_0^t s^n e^{-ws}dF_u\left(s\right)}{\int_0^t s^n e^{-ws}dF_v\left(s\right)} \geq \frac{\int_0^\infty s^n e^{-ws}dF_u\left(s\right)}{\int_0^\infty s^n e^{-ws}dF_v\left(s\right)}.$$

Since

$$\frac{\int_0^\infty s^n e^{-ws}dF_u\left(s\right)}{\int_0^\infty s^n e^{-ws}dF_v\left(s\right)} = \lim_{t\to\infty} \frac{\int_0^t s^n e^{-ws}dF_u\left(s\right)}{\int_0^t s^n e^{-ws}dF_v\left(s\right)},$$

the result will be done if we prove that

$$K\left(t\right) = \frac{\int_0^t s^n e^{-ws}dF_u\left(s\right)}{\int_0^t s^n e^{-ws}dF_v\left(s\right)}$$

[1] See Lemma 1, page 148 in Feller, 1971.

is decreasing in t.

A simple calculation shows that $K(.)$ is decreasing if and only if

$$\frac{dF_u(t)}{dF_v(t)} \leq K(t). \tag{4}$$

We now prove (4):

$$\begin{aligned} K(t) &= \frac{\int_0^t s^n e^{-ws} \frac{dF_u(s)}{dF_v(s)} dF_v(s)}{\int_0^t s^n e^{-ws} dF_v(s)} \\ &\geq \frac{dF_u(t)}{dF_v(t)} \frac{\int_0^t s^n e^{-ws} dF_v(s)}{\int_0^t s^n e^{-ws} dF_v(s)} = \frac{dF_u(t)}{dF_v(t)}. \end{aligned}$$

This completes the proof of the theorem. ■

4 Concluding comments

Up to now it was obtained in the literature on restrictions on utility functions (Dionne and Eeckhoudt, 1985, Briys and Schlesinger, 1990 and Sweeny and Beard, 1992) that more risk averse individuals do not necessarily produce more self-protection activities or are nor willing to pay more for a lower probability of death. This counter intuitive result was explained by the fact that reducing the probability of accident is not limited to a pure second order effect but introduces a first order shift as well as other moments shifts. Comparative mixed risk aversion is shown to be very useful for making comparisons of self-protection activities and willingness to pay for a lower probability of accident or death by Dachraoui et al. [2000].

By using the concept of comparative mixed risk aversion defined in this note, Dachraoui et al. [2000] show that more mixed risk averse individuals have a Proper Risk Behavior in the sense that they choose a higher level of self-protection or are willing to pay more for a lower probability of accident when the probability of accident is lower than 1/2.

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