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Abstract

In this note we analyze the hedging property of an optimal portfolio with one risk-free asset and two risky assets. We make a restriction on the dependence between the two risky assets and show that the sign of the covariance is necessary and sufficient to set the relative investments in the two risky assets of the portfolio for all concave utility functions. One application of our result concerns derivatives with linear payoffs. We also show how our model is related to the mutual fund separation condition proposed by Ross (1978).

Keywords: Optimal financial portfolio, covariance, separation condition, regression dependence.

Résumé

Dans cette note, nous analysons les propriétés de diversification d'un portefeuille optimal composé d'un actif sans risque et de deux actifs risqués. Nous introduisons une restriction sur la dépendance entre les deux actifs risqués et nous montrons que le signe de la covariance est une condition nécessaire et suffisante pour fixer les investissements relatifs dans les deux titres pour toutes les fonctions d'utilité concaves. Une application de notre résultat est la composition de portefeuilles avec produits dérivés. Nous montrons également que notre modèle est directement relié à la condition de séparation des portefeuilles proposée par Ross (1978).

Mots clés : Portefeuille financier optimal, covariance, condition de séparation, dépendance par régression.

1 Introduction

Contrarily to many insurable risks where, by the law of large numbers, pooling is sufficient to reduce risk, financial risks which usually involve correlated risks, cannot be eliminated through pooling. Financial risk reduction is based on a simple mechanism: create a random variable correlated with the risky assets. The returns on these securities are usually highly positively correlated or highly negatively correlated with the basic assets. In each case, it is possible to obtain risk reduction by taking an appropriate position in the derivative instrument. The intuition suggests that if the correlation is positive we take a short position and if it is negative we take a long position on the derivative. While this mechanism is widely accepted, its economic foundation has still to rely on mean-variance analysis.

In this note we study the composition of optimal portfolios by exploring the relation between the relative investments in risky assets and the covariance of distribution returns under risk aversion. In finance theory, the study of optimal financial portfolios is still an open question. Without strong assumptions on either utility or distributions functions, few results are available. It is even difficult to link the optimal shares of risky assets to a simple measure of statistical dependence out of mean-variance preferences. However, the intuitive result that $Sign\,(\alpha_1^*\alpha_2^*) = -Sign\,(\sigma_{12})$ under risk aversion (where α_1^* and α_2^* are the investment in risky assets $\widetilde{x}_1$ and $\widetilde{x}_2$ respectively and σ_{12} is their covariance) would be useful for the management of many portfolios since it is always more easy to estimate the parameters of assets distributions than those of utility functions.

In this note, we introduce the concept of Regression Dependence (Lehman, 1966 and Tukey, 1958) and set the relation of the relative investments in the two risky assets to the covariance of both assets without any restriction on preferences. In the last section, we provide examples of regression dependent distributions that are related to economic and finance applications.

2 Optimal portfolio

In the standard problem of portfolio choice, where a risk averse investor is allocating his wealth between a risk-free asset and a risky asset, it is well known that the optimal position to take on the risky asset (long vs short) is only function of its excess expected return on the risk-free asset. When we consider a potential portfolio of two risky assets and a risk-free asset in the model, the optimal position to take on one asset is no longer function only of its expected excess return, but it depends also on the correlation between the risky assets, which is the hedging effect. If, for example, the two risky assets are negatively correlated, it may be optimal to hold both in positive quantities, even if one risky asset offers no risk premium.

In our model we consider a risk averse agent who allocates his wealth (normalized to one) between one risk-free asset (with return x_0) and two risky assets with returns $\tilde{x}_i$ for $i = 1, 2$. We note α_i, $i = 1, 2$ as the investment in asset $\widetilde{x}_i$. The agent's end of period wealth W is then equal to

$$W(\alpha_1, \alpha_2) = 1 + x_0 + \alpha_1 (x_1 - x_0) + \alpha_2(x_2 - x_0),$$

The optimal portfolio solves the following program (P):

$$\max_{\alpha_1,\alpha_2} \int_{\underline{x}_1}^{\overline{x}_1} \int_{\underline{x}_2}^{\overline{x}_2} u\left(W(\alpha_1, \alpha_2)\right) dF\left(x_1/x_2\right) dG\left(x_2\right)$$

where $[\underline{x}_1, \overline{x}_1]$ and $[\underline{x}_2, \overline{x}_2]$ are respectively the support of $\tilde{x}_1$ and $\tilde{x}_2$.

Assume we have interior solutions, the first order conditions associated to the above problem are:

$$\int_{\underline{x}_1}^{\overline{x}_1} \int_{\underline{x}_2}^{\overline{x}_2} (x_1 - x_0)\, u^{'}\left(W(\alpha_1, \alpha_2)\right) dF\left(x_1/x_2\right) dG\left(x_2\right) = 0, \qquad (1)$$

$$\int_{\underline{x}_1}^{\overline{x}_1} \int_{\underline{x}_2}^{\overline{x}_2} (x_2 - x_0)\, u^{'}\left(W(\alpha_1, \alpha_2)\right) dF\left(x_1/x_2\right) dG\left(x_2\right) = 0. \tag{2}$$

The above equations are necessary and sufficient conditions for an optimal solution under strict risk aversion or when u is strictly concave.

In the case of independence between the risky assets the first order condition (1) evaluated at $\alpha_1 = 0$ can be written as

$$\left(E\left(\widetilde{x}_1\right) - x_0\right) \int_{\underline{x}_2}^{\overline{x}_2} u^{'}\left(\alpha_2\left(x_2 - x_0\right) + 1 + x_0\right) dG\left(x_2\right)$$

which has the sign of the expected excess return $\left(E\left(\widetilde{x}_1\right) - x_0 \equiv m_1\right)$. It follows that α_1^* is positive if and only if m_1 is also positive, that is if and only if $\widetilde{x}_1$ offers a risk premium. The same logic applies for α_2^*.

To illustrate the difficulty of allowing m_2 to be non nil in the reminder of the paper, we take the example of the quadratic utility function. The above first order conditions associated to the optimal portfolio become:

$$\begin{aligned} \alpha_1\left(\sigma_{11} + m_1^2\right) + \alpha_2\left(\sigma_{12} + m_1 m_2\right) &= m_1 \\ \alpha_2\left(\sigma_{22} + m_2^2\right) + \alpha_1\left(\sigma_{12} + m_1 m_2\right) &= m_2 \end{aligned}$$

where σ_{11} and σ_{22} stand for the variances of the risky assets, σ_{12} is for the covariance, and m_1 (respectively m_2) is the excess return over the risk-free rate of asset 1 (respectively asset 2).

The solution of the above system of equations yields:

$$\begin{aligned} \alpha_1^* &= \frac{m_1\sigma_{22} - m_2\sigma_{12}}{m_1^2\sigma_{22} - \sigma_{12}^2 + \sigma_{11}\sigma_{22} + m_2^2\sigma_{11} - 2m_1m_2\sigma_{12}} \\ \alpha_2^* &= \frac{m_2\sigma_{11} - m_1\sigma_{12}}{m_1^2\sigma_{22} - \sigma_{12}^2 + \sigma_{11}\sigma_{22} + m_2^2\sigma_{11} - 2m_1m_2\sigma_{12}}. \end{aligned}$$

It is easily seen that the values of the α_i^* are driven by their respective m_i, as for independence, and by the hedging effect measured by σ_{12}. We observe, for example, that for the case of negative correlation and $m_1 > 0$, both α_i^*

can be positive even if m_2 is negative. In situations where risky assets are positively correlated it may be optimal to take a short position in $\widetilde{x}_2$ even if m_2 is positive. So, many solutions are possible for a given sign of the covariance (σ_{12}) and $m_2 \neq 0$. Consequently, in order to isolate the hedging effect and eliminate that of the expected return m_2 we have to restrict m_2 to be nil. This actuarially fair asset can be interpreted as a derivative product used to hedge the risk on $\widetilde{x}_1$.

As we discussed earlier even when $m_2 = 0$, the optimal asset proportion α_2^* is not equal to zero, the reason is that this asset can be used to hedge the optimal portfolio. In fact, we are going to show that $Sign\,(\alpha_1^* \alpha_2^*) = -Sign\,(\sigma_{12})$ for a particular assumption on the stochastic dependence between the two risky assets, which helps to reduce the volatility of the portfolio.

We now introduce the next definition:

Definition 1 *(Tukey, 1958) We say that the distribution of $\widetilde{x}_1$ given $\widetilde{x}_2$ shows complete negative regression on $\widetilde{x}_2$ if the cumulative distribution $F(x_1/x_2)$ satisfies*

$$F(x_1/x_2') \leq F(x_1/x_2) \text{ for } x_2' \leq x_2.$$

We define complete positive regression analogously.

Note that complete regression guarantees that if the covariance is nil then the two random variables are independent, and if $\widetilde{x}_1$ given $\widetilde{x}_2$ shows complete negative (respectively positive) regression on $\widetilde{x}_2$ then the covariance between $\widetilde{x}_1$ and $\widetilde{x}_2$ is negative (respectively positive). We have the next result:

Proposition 1 *Suppose $m_2 = 0$ and let $\widetilde{x}_1$ shows either complete negative regression or complete positive regression on $\widetilde{x}_2$, then*

$$Sign\,(\alpha_1^* \alpha_2^*) = -Sign\,(Cov\,(\widetilde{x}_1, \widetilde{x}_2))$$

and

$$\alpha_2^* = 0 \text{ if and only if } Cov\left(\widetilde{x}_1, \widetilde{x}_2\right) = 0,$$

for all concave utility functions.

Proof.

From the first order condition (2) we can write

$$Eu_{\alpha_2} = \int_{\underline{x}_2}^{\overline{x}_2} \left(x_2 - x_0\right) I\left(x_2\right) dG\left(x_2\right) = 0$$

where

$$I\left(x_2\right) = \int_{\underline{x}_1}^{\overline{x}_1} u'\left(W(\alpha_1^*, \alpha_2^*)\right) dF\left(x_1/x_2\right).$$

Or, by integrating by part,

$$\begin{aligned} Eu_{\alpha_2} &= m_2 I\left(\overline{x}_2\right) - \int_{\underline{x}_2}^{\overline{x}_2} \left[\int_{\underline{x}_2}^{x_2} \left(u - x_0\right) dG\left(u\right)\right] I'\left(x_2\right) dx_2 \\ &= -\int_{\underline{x}_2}^{\overline{x}_2} \left[\int_{\underline{x}_2}^{x_2} \left(u - x_0\right) dG\left(u\right)\right] I'\left(x_2\right) dx_2. \end{aligned}$$

Since $m_2 = 0$, it follows that

$$\int_{\underline{x}_2}^{x_2} \left(u - x_0\right) dG\left(u\right) \leq 0.$$

Consequently, if $I\left(.\right)$ is monotonic then $Sign(Eu_{\alpha_2}) = Sign(I'\left(.\right))$. The first order condition that guarantees an interior solution will be then violated in this case.

An integration by part of the first derivative of $I\left(x_2\right)$ gives

$$\begin{aligned} I'\left(x_2\right) &= \alpha_2^* \int_{\underline{x}_1}^{\overline{x}_1} u''\left(W(\alpha_1^*, \alpha_2^*)\right) dF\left(x_1/x_2\right) dx_1 \\ &\quad -\alpha_1^* \int_{\underline{x}_1}^{\overline{x}_1} u''\left(W(\alpha_1^*, \alpha_2^*)\right) \left[\int_{\underline{x}_1}^{x_1} \frac{\partial}{\partial x_2} F\left(t/x_2\right)\right] dx_1, \end{aligned}$$

which can be rewritten as

$$
\begin{aligned}
I^{'}\left(x_{2}\right) &= \alpha_{2}^{*}\int_{\underline{x}_{1}}^{\overline{x}_{1}} u''\left(W(\alpha_{1}^{*},\alpha_{2}^{*})\right) dF\left(x_{1}/x_{2}\right) dx_{1} \\
&\quad -\alpha_{1}^{*}\int_{\underline{x}_{1}}^{\overline{x}_{1}} u''\left(W(\alpha_{1}^{*},\alpha_{2}^{*})\right) \frac{\partial}{\partial x_{2}} F\left(x_{1}/x_{2}\right) dx_{1} \\
&= \alpha_{2}^{*}[\int_{\underline{x}_{1}}^{\overline{x}_{1}} u''\left(W(\alpha_{1}^{*},\alpha_{2}^{*})\right) dF\left(x_{1}/x_{2}\right) dx_{1} \\
&\quad -\frac{\alpha_{1}^{*}}{\alpha_{2}^{*}}\int_{\underline{x}_{1}}^{\overline{x}_{1}} u''\left(W(\alpha_{1}^{*},\alpha_{2}^{*})\right) \frac{\partial}{\partial x_{2}} F\left(x_{1}/x_{2}\right) dx_{1}].
\end{aligned}
$$

Consequently, a necessary condition for $I\left(.\right)$ to be non monotonic is that

$$
Sign\left(\alpha_{1}^{*}\alpha_{2}^{*}\right) = Sign\left(\frac{\partial}{\partial x_{2}}F\left(./x_{2}\right)\right). \tag{3}
$$

In fact, suppose (3) is not true, then we will have

$$
Sign\left(\alpha_{1}^{*}\alpha_{2}^{*}\right) = -Sign\left(\frac{\partial}{\partial x_{2}}F\left(./x_{2}\right)\right)
$$

and $Sign\left(I^{'}\left(x_{2}\right)\right) = Sign(\alpha_{2}^{*}u'')$ making $I\left(.\right)$ a monotone function. As discussed earlier this should not be the case if we impose an interior solution.

Under our assumption of regression dependence, we have that

$$
Sign\left(cov\left(\widetilde{x}_{1},\widetilde{x}_{2}\right)\right) = -Sign\left(\frac{\partial}{\partial x_{2}}F\left(./x_{2}\right)\right). \tag{4}
$$

(3) and (4) end the proof of the first part of Proposition 1.

We now show the second part of Proposition 1. We know that if $\widetilde{x}_{1}$ and $\widetilde{x}_{2}$ are independent then $\alpha_{2}^{*}=0$. It remains to show that if $\alpha_{2}^{*}=0$ then the two random variables are independent. We evaluate the first order condition (2) at $\alpha_{2}=0$, and obtain

$$
Eu_{\alpha_{2}} = \int_{\underline{x}_{2}}^{\overline{x}_{2}} \left(x_{2}-x_{0}\right)\left[\int_{\underline{x}_{1}}^{\overline{x}_{1}} u'\left(W(\alpha_{1}^{*},0)\right) dF\left(x_{1}/x_{2}\right)\right] dG\left(x_{2}\right)
$$

$$= -\int_{\underline{x}_2}^{\overline{x}_2} \left[\int_{\underline{x}_2}^{x_2} (u - x_0)\, dG(u) \right] \frac{\partial}{\partial x_2} \left[\int_{\underline{x}_1}^{\overline{x}_1} u'(W(\alpha_1^*, 0))\, dF(x_1/x_2) \right] dx_2$$

$$= \alpha_1^* \int_{\underline{x}_1}^{\overline{x}_1} \int_{\underline{x}_2}^{\overline{x}_2} \left[\int_{\underline{x}_2}^{x_2} (u - x_0)\, dG(u) \right] u''(W(\alpha_1^*, 0)) \frac{\partial}{\partial x_2} F(x_1/x_2)\, dx_1 dx_2.$$

The preceding equality is obtained following an integration by part. Under our assumptions of dependence regression, risk aversion, and $m_2 = 0$, for Eu_{α_2} evaluated at $\alpha_2 = 0$ to be nil, we need to have

$$\frac{\partial}{\partial x_2} F(./x_2) = 0 \text{ for all } x_2$$

which means that $\tilde{x}_1$ and $\tilde{x}_2$ should be independent. *Q.E.D.*

Proposition 1 shows that even if the second asset is actuarially fair ($m_2 = 0$) the asset proportion α_2^* at the optimum is not trivially equal to zero since it can be used for hedging purposes when $\widetilde{x}_2$ is correlated with $\widetilde{x}_1$. If the two assets are not correlated, then a risk averse investor would not invest in the second asset which confirms the existence of hedging in the optimal portfolio. This conclusion is confirmed by the relation through the covariance of the two random variables.

Now we turn to identify the different positions (long vs short) that the investor takes on the first risky asset depending on the expected return. We know that, in the situation where an agent is allocating his wealth between a risk-free asset and a risky asset, a necessary and a sufficient condition for an agent to invest a positive amount in the risky asset is that the expected return exceeds that of the riskless asset. In the next proposition we generalize this result to our model. In fact we can prove the next result:

Proposition 2 *Suppose $m_2 = 0$, then a necessary and a sufficient condition for the risk averse agent to invest a positive amount in the risky asset $\widetilde{x}_1$ is that the expected return on this asset exceeds that of the riskless asset (x_0).*

Proof.

We need to prove that

$$Sign\left(\alpha_1^*\right) = Sign\left(m_1\right),$$

or equivalently that the first order condition (1) evaluated at $\alpha_1 = 0$, has the same sign as m_1. When $\alpha_1 = 0$, we verify that (2) is reduced to

$$Eu_{\alpha_2}\left(0, \alpha_2\right) = \int_{\underline{x}_2}^{\overline{x}_2} \left(x_2 - x_0\right) u'\left(\alpha_2\left(x_2 - x_0\right)\right) dG\left(x_2\right)$$

which implies that

$$Sign\left(Eu_{\alpha_2}\left(0, \alpha_2\right)\right) = -Sign\left(\alpha_2\right).$$

By the above expression we see that if the individual invests 0 in the first asset then he would invest 0 in the second asset. It is consequently sufficient to prove that $Eu_{\alpha_1}(0, 0)$ has the same sign as m_1.

From the first order condition (1),

$$Eu_{\alpha_1}\left(0, 0\right) = u'\left(0\right) m_1.$$

Since $u'() \geq 0$, by the concavity of u we have that $Sign\left(\alpha_1^*\right) = Sign\left(m_1\right)$.

Q.E.D.

3 Examples

We now discuss examples of regression dependence (for other examples see Lehman, 1966).

Example 1. Let $Y = \alpha + \beta X + U$, where X and U are independent. Then Y is positively or negatively regression dependent on X as $\beta \geq 0$ or ≤ 0. In fact, the conditional distribution of Y given $X = x$ is that of $\alpha + \beta x + U$ and hence is stochastically increasing in x if $\beta \geq 0$. In particular, the components of a bivariate normal distributions are positively or negatively regression dependent as the correlation coefficient is positive or negative (see the Appendix for a formal proof). Many other distributions satisfy the above relation between Y and X.

Example 2. The second example is related to the set of distributions that allow for two-mutual funds separation as stressed by Ross [1978]. We now show that the distributions allowing for two-mutual funds separation are in the class of regression dependent distributions. In fact, suppose that our two risky asset distributions allow for two-mutual funds separation. We know from Ross [1978] that a necessary condition is that $\widetilde{x}_1$ and $\widetilde{x}_2$ can be written as

$$\widetilde{x}_i = x_0 + \beta_i(\widetilde{Y} - x_0) + \widetilde{\varepsilon}_i, \text{ for i=1,2.}$$

The conditional distribution function can now be written as

$$F(x_1/x_2) = Pr\left(\widetilde{\varepsilon}_1 - \tfrac{\beta_1}{\beta_2}\widetilde{\varepsilon}_2 \leq x_1 - x_0 - \tfrac{\beta_1}{\beta_2}(x_2 - x_0)\right).$$

As we can see, $F(x_1/x_2)$ is always monotone in x_2 and the sign of this monotonicity depends on those of β_1 and β_2 which represent the sensitivity of each risky asset with $\widetilde{Y}$. $\widetilde{Y}$ can be interpreted as the market risk or any other index. In fact, we have

$$Sign(\frac{\partial}{\partial x_2}F\left(./x_2\right)) = -Sign(\beta_1\beta_2),$$

which yields, if we apply the result in Proposition 1,

$$Sign\left(\alpha_1^*\alpha_2^*\right) = Sign(\beta_1\beta_2).$$

4 Conclusion

In this note we propose an economic rational of hedging for all risk averse utility functions. We have shown that financial risk of a risky asset can be reduced by combining it with a negative correlated asset or derivative. While we restricted the dependence of the risky assets to regression dependence, we were able to extend the result used in finance in the class of mean-variance preferences to all risk averse agents. This work joins that of Clark and Jokung [1999] in analyzing the composition of the optimal portfolio and where restrictions are made on distributions returns rather than on preferences.

5 Appendix: bivariate normal distribution and dependence regression

Let us consider the bivariate normal distribution

$$f(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{q}{2}}, \quad -\infty < x_1,\ x_2 < \infty$$

where $\sigma_1,\ \sigma_2 > 0$, $-1 < \rho < 1$ and

$$q = \frac{1}{1-\rho^2}[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right) + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2].$$

The conditional p.d.f. of $\widetilde{x}_1$ given $\widetilde{x}_2 = x_2$, is itself normally distributed with mean

$$\mu = \mu_1 + \rho\frac{\sigma_1}{\sigma_2}(x_2 - \mu_2)$$

and variance

$$\sigma^2 = \sigma_1^2\left(1 - \rho^2\right).$$

Thus, with a bivariate normal distribution, the conditional distributions function of $\widetilde{x}_1$ given x_2 is given by

$$F(x_1/x_2) = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{x_1} e^{-\frac{1}{2}\left(\frac{u-\mu}{\sigma}\right)^2} du.$$

Taking the derivative with respect to x_2 yields

$$\frac{\partial}{\partial x_2}F(x_1/x_2) = \frac{1}{\sqrt{2\pi}\sigma}\rho\frac{\sigma_1}{\sigma_2}\int_{-\infty}^{x_1}\left(\frac{u-\mu}{\sigma^2}\right) e^{-\frac{1}{2}\left(\frac{u-\mu}{\sigma}\right)^2} du.$$

Simple calculus shows that

$$\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{x_1}\left(\frac{u-\mu}{\sigma^2}\right)e^{-\frac{1}{2}\left(\frac{u-\mu}{\sigma}\right)^2}du \leq 0.$$

It follows that

$$Sign(\frac{\partial}{\partial x_2}F\left(x_1/x_2\right)) = -Sign\left(\rho\right).$$

The dependence regression is then verified and is determined by the sign of the correlation between $\widetilde{x}_1$ and $\widetilde{x}_2$.

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