Conditions Ensuring the Decomposition of Asset Demand for All Risk-Averse Investors[1]

by Kaïs Dachraoui and Georges Dionne

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Abstract

We explore how the demand for a risky asset can be decomposed into an investment effect and a hedging effect by all risk-averse investors. This question has been shown to be complex when considered outside of the mean-variance framework. We restrict dependence among returns on the risky assets to quadrant dependence and find that the demand for one risky asset can be decomposed into an investment component based on the risk premium offered by the asset and a hedging component used against the fluctuations in the return on the other risky asset. We also discuss how the class of quadrant dependent distributions is related to that of two-fund separating distributions. This contribution opens up the search for broader distributional hypotheses suitable to asset demand models. Examples are discussed.

Keywords: Portfolio choice, investment effect, hedging effect, quadrant dependence, two-fund separation, asset demand model.

JEL classification: D80, G10, G11, G12.

1. Introduction

The mean-variance model of portfolio choice has been used extensively to answer the following question: Under what conditions can the demand for one risky asset be decomposed into an investment part and a hedging part? (Mossin, 1973; Huang and Litzenberger, 1988). Though commonly used, the mean-variance model imposes strong conditions either on preferences or on return distributions (i.e. quadratic utility function or elliptical distributions). The class of elliptical distributions contains the multivariate normal distribution as well the multivariate *t*-distribution as special cases (Owen and Robinovitch, 1983). The normal distribution has been challenged by many empirical studies (Zhou, 1993) and the quadratic utility function displays increasing absolute risk aversion. More recently, Beaulieu, Dufour and Khalaf (2005) have shown that the mean-variance framework (or models limited to two-parameter distributions) is still rejected (though less frequently) when non-normal distributions such as the *t*-distribution are considered. They concluded that more research is needed to better identify the distribution hypotheses applicable to asset demand models.

The main objective of this article is to provide conditions ensuring the decomposition of asset demand for all risk-averse investors. It also proposes a class of distribution functions which differs from the two-fund separation distributions. We use a different form of risk dependence, namely quadrant dependence. This concept has been defined by Lehmann (1966). This form of non-linear dependence describes how two random variables behave together when they are simultaneously small (or large). One important property of quadrant dependence is that if $(\mathrm{R}_1, \mathrm{R}_2)$ is positive (negative) quadrant dependent, then the covariance between R_1 and R_2 is positive (negative). However, the converse is not true (Tong, 1980).

Quadrant dependence has its interest in modeling dependent risks since it can take into account the simultaneous downside (upside) evolution of asset prices by introducing a natural hedging property. Quadrant dependence is of particular interest in risk management since it looks at the joint occurrence of large losses. In this article we shall show how quadrant dependence permits the decomposition of asset demand in a very natural manner. Our results open up the search for more general asset demand models that can free the use of stochastic dependence from its connection with linear correlation.

The two-fund separation theorem of Ross (1978) is associated to elliptical distributions (Owen and Rabinovitch, 1983; Chamberlain, 1983; Elton and Gruber, 2000) for all concave and increasing utility functions while that of Cass and Stiglitz (1970) is associated to utility functions satisfying marginal utility conditions (HARA) for all distribution functions. In this article, we are limited to conditions on distribution functions. Though elliptical distributions imply separation, the converse may not be true. We shall discuss how the Ross mutual fund separation theorem is related to the family of quadrant dependent distributions. We shall also provide an example of a joint distribution in the quadrant dependent family that is not a two-fund separating distribution. These results indicate that the class of quadrant dependent distributions differs from that of two-fund separating distributions. However, we do not examine conditions on distributions to obtain separation according to Ross (1978).

Section 2 presents our model of portfolio choice and introduces the concept of quadrant dependence (Lehmann, 1966). In this section, we also derive our main results related to the decomposition of portfolio demand into an investment part and a hedging part. In Section 3, we establish formal links between quadrant dependence and mutual fund separation. We also provide examples of quadrant dependent distributions for applications in finance. Section 4 concludes the article. All proofs are available in the Appendix at: http://neumann.hec.ca/gestiondesrisques/04-01.pdf.

2. Characterizing optimal portfolios

2.1 Basic Model

We consider a risk-averse agent who allocates his wealth (normalized to one) between one risk-free asset (with return r_0) and two risky assets with returns R_i, for i = 1,2. We denote the joint distribution function as $F(r_1, r_2)$. We also note $\left[\underline{r}_1, \overline{r}_1\right]$ and $\left[\underline{r}_2, \overline{r}_2\right]$ as the supports for R_1 and R_2, respectively, and $\alpha_i, i = 0,1,2$, as the demand for asset *i* chosen so as to maximize expected utility in a world with unlimited short-selling and under the constraint that $\alpha_0 + \alpha_1 + \alpha_2 = 1$. The agent's random end-of-period wealth W is then equal to

$$W\left(\alpha_1, \alpha_2\right) = 1 + r_0 + \alpha_1\left(R_1 - r_0\right) + \alpha_2\left(R_2 - r_0\right). \tag{1}$$

We define *E* as the expectation operator and m_i as the risk premium associated with asset *i*, that is $m_i = E\left(R_i\right) - r_0$, for i = 1,2. As usual, the individual has a von-Neumann-Morgenstern utility function, $u(.)$, which we assume to be increasing, concave in final wealth, and continuously differentiable to the second order. This last assumption is for convenience and is not necessary to derive our results. So the optimal portfolio is obtained by maximizing $E\left(u\left(W\left(\alpha_1, \alpha_2\right)\right)\right)$ with respect to α_1 and α_2.

In the case of independence among the risky assets, the first derivative of the expected utility with respect to α_1, $\left(E\left(u'\left(W\left(\alpha_1, \alpha_2\right)\right)\left(R_1 - r_0\right)\right) = 0\right)$, evaluated at $\alpha_1 = 0$, can be written as

$$\left(E\left(R_1\right) - r_0\right)\ E\left(u'\left(1 + r_0 + \alpha_2\left(R_2 - r_0\right)\right)\right), \tag{2}$$

which has the sign of the risk premium associated with R_1. It follows that α_1^* is positive, if and only if m_1 is positive, that is if and only if R_1 offers a positive risk premium. The same logic applies to α_2^*. Allowing for dependence among returns on risky assets will make it more difficult to characterize the optimal portfolio. As an illustration, we consider, for a moment, the case of mean-variance preferences; to be precise, we suppose $u(W) = W - \frac{b}{2}W^2$, where *b* is a positive parameter that captures the agent's risk aversion. We also assume the following regularity condition on the first derivative $u'(W) = 1 - bW > 0$ for all W. The explicit solution to the maximization problem yields :

$$\alpha_1^* = \frac{1 - b(1+r_0)}{b} \frac{m_1\sigma_{22} - m_2\sigma_{12}}{\Delta}, \tag{3a}$$

$$\alpha_2^* = \frac{1 - b(1+r_0)}{b} \frac{m_2\sigma_{11} - m_1\sigma_{12}}{\Delta}, \tag{3b}$$

where $\sigma_{ij} = Cov(R_i, R_j)$, $1 - b(1+r_0) > 0$ from the regularity condition and $\Delta = m_2^2\sigma_{11} + m_1^2\sigma_{22} - 2m_1m_2\sigma_{12} + \sigma_{11}\sigma_{22} - \sigma_{12}^2 > 0$, since the covariance matrix is positive semi-definite (see the Appendix for details). It is easily observed that α_2^*, the optimal demand of asset 2, is a function of m_2, σ_{11}, and of σ_{12} as well as a function of *b* and m_1. For given risk aversion (*b*) and risk premium for asset 1 (m_1), the demand for asset 2 can be decomposed into an investment part $\left(\alpha_{2m}^*\right)$ related to the mean-return of the asset and the hedging part $\left(\alpha_{2h}^*\right)$ related to its diversification aspect.

So α_2^* can be decomposed into

$$\alpha_{2h}^{*} = -\frac{\sigma_{12}}{\sigma_{22}}\alpha_{1}^{*} = -\frac{1-b(1+r_0)}{b}\frac{\sigma_{12}}{\sigma_{22}}\frac{m_1\sigma_{22} - m_2\sigma_{12}}{\Delta}, \tag{4a}$$

the hedging part $\left(-\frac{\sigma_{12}}{\sigma_{22}}\alpha_{1}^{*}\right)$, and

$$\alpha_{2m}^{*} = \frac{1-b(1+r_0)}{b}\frac{1}{\sigma_{22}}\frac{\sigma_{11}\sigma_{22} - (\sigma_{12})^2}{\Delta}m_2 = km_2, \tag{4b}$$

the investment part (km_2). The sign of investment part (km_2) depends on the risk premium (m_2) offered by the risky asset, and the sign of the hedging part $\left(-\frac{\sigma_{12}}{\sigma_{22}}\alpha_{1}^{*}\right)$ is a function of the covariance between the returns of the two assets (σ_{12}). Since *k* is strictly positive (b, Δ and σ_{22} are positive, $\sigma_{11}\sigma_{12} - (\sigma_{12})^2 > 0$, a property of the covariance matrix, and $1-b(1+r_0) > 0$ by definition of marginal utility), $\alpha_{2m}^{*} = \alpha_{2}^{*} - \alpha_{2h}^{*}$ is proportional to m_2. Moreover, from the hedging part, $\mathrm{Sign}(\alpha_{1}^{*}\alpha_{2h}^{*}) = -\mathrm{Sign}(\sigma_{12})$ since $\sigma_{22} > 0$. In the next section, we show how the set of return distributions proposed in this article can be used to obtain such decomposition for all risk-averse investors.

2.2 Quadrant Dependence

The notion of quadrant dependence is used for the comparison of the probability of any quadrant $R_1 \le r_1$, $R_2 \le r_2$ under a given distribution of (R_1, R_2) with the corresponding probability in the case of independence. More formally, we have the next definition.

Definition 1 *(Lehmann, 1966): Let* (R_1, R_2) *be a bivariate random variable. We say that* (R_1, R_2) *is positively quadrant dependent (PQD, in short) if*

$$P(R_1 \le r_1, R_2 \le r_2) \ge P(R_1 \le r_1)P(R_2 \le r_2) \text{ for all } r_1,\ r_2. \tag{5}$$

The dependence is strict if inequality holds for at least some pair (r_1, r_2). Similarly, (R_1, R_2) is negatively quadrant dependent if (5) holds with the inequality sign reversed. Intuitively, R_1 and R_2 are PQD if the probability that they are simultaneously small (or simultaneously large) is at least as great as it would be were they independent. PQD is invariant under strictly increasing transformations of the random variables.

Equation (5) can be equivalently written as

$$P(R_1 \le r_1 \,|\, R_2 \le r_2) \ge P(R_1 \le r_1) \text{ for all } r_1,\ r_2. \tag{6}$$

Under this form, PQD expresses the fact that knowledge of R_2 being small increases the probability of R_1 being small. PQD is satisfied when random variables are associated that is when $\mathrm{Cov}\left(g_1(R_1), g_2(R_2)\right) \ge 0$ holds for all non-decreasing real-valued functions g_1 and g_2, and where Cov means covariance of $g_1(R_1)$ and $g_2(R_2)$ (see Milgrom and Weber, 1982, for application to auction theory). PQD is also fulfilled if (R_1, R_2) shows positive likelihood ratio dependence (PLRD, in short. See Lehmann, 1966). PLRD is obtained by requiring that the conditional density of R_1, given R_2, is monotonic. In other words, the random variable (R_1, R_2) or its distribution is PLRD if

$$f\left(r_1, r_2^*\right) f\left(r_1^* r_2\right) \leq f\left(r_1, r_2\right) f\left(r_1^* r_2^*\right)$$

holds for all $r_1 > r_1^*$ and $r_2 > r_2^*$ (Tong, 1980, p. 79). This means that the likehood is larger when coordinates take larger values together and smaller values together at the same time. The bivariate normal density is an example of PLRD.

Under the assumption of quadrant dependence, we are able to establish our main result:

Proposition 1: *Let* (R_1, R_2) *be quadrant dependent, and let* $\left(\alpha_1^*, \alpha_2^*\right)$ *be the optimal portfolio, then* α_i^* *can be decomposed for all risk-averse investors as* $\alpha_i^* = \alpha_{im}^* + \alpha_{ih}^*$*, for* $i = 1,\ 2$ *with*

a) $\alpha_{im}^* \geq 0$ *if and only if* $E\left(R_i\right) \geq r_0$*, and*

b) $\text{Sign}\left(\alpha_j^* \alpha_{ih}^*\right) = -\text{Sign}\left(\text{Cov}\left(R_1, R_2\right)\right)$*, for* $j \neq i$.

Proof: See the Appendix.

As before, α_{im}^* and α_{ih}^*, for i = 1,2, designate respectively, the investment part and the hedging part of asset *i* demand. It should be noted that the decomposition of the optimal portfolio is investor-specific, as for the quadratic utility function in the preceding section, where α_{2h}^* and α_{2m}^* in (4) are function of the parameter *b* that captures risk-aversion. However, the above conditions a) and b) do not yield explicit values for α_{im}^* and α_{ih}^*. In that sense they are more vaguely characterized than with the quadratic utility function. The intuition behind Proposition 1 is natural and a significant implication of the proposition is that we need only know the sign of the covariance and that of the risk premium to sign the hedging effect and the investment effect, respectively, even if we do not restrict our analysis to the mean-variance model.

One corollary from Proposition 1 is that the optimal positions (long vs. short) on the investment component (α_{im}^*) and the hedging component (α_{ih}^*) will depend solely on the distributions of the risky assets for all risk-averse investors. Preferences determine the trade-off between the investment component and the hedging component and set the total demand for the risky asset. In the next proposition we look at the situations where one asset has a zero risk premium or where the assets returns are not correlated. We have the next result.

Proposition 2: *Let* (R_1, R_2) *be quadrant dependent, then for all risk-averse investors and* i=1,2*:*

$\alpha_{ih}^* = 0$ *if and only if* $\mathrm{Cov}(R_1, R_2) = 0$ *, and*

$\alpha_{im}^* = 0$ *if and only if* $E(R_i) - r_0 = 0$.

Proof: See the Appendix.

In the particular case where one risky asset has a zero risk premium, Proposition 2 shows that a risk-averse investor may invest money in a risky asset even though there is no risk premium attached. The reason is that the financial risk can be reduced by investing in a correlated risky asset. The returns on this security may display either a strong positive or negative correlation with the basic asset. This result, already known for a mean-variance model, is extended in Proposition 2 to all risk-averse investors when (R_1, R_2) is quadrant dependent. To complete the characterization of the optimal financial portfolio, we now proceed to identify the different positions (long vs. short) that the investor will take on one risky asset if the other risky asset has a zero risk premium. As we already know, in a mean-variance context, when an agent is allocating his wealth between a risk-free asset and one risky asset or when the two risky assets have independent returns, a positive risk premium is necessary and sufficient to obtain a positive investment. In the next proposition we generalize this result to all risk-averse investors when (R_1, R_2) is quadrant dependent.

Proposition 3: *Let* (R_1, R_2) *be quadrant dependent. If* $m_j = 0$, $j = 1, 2$, *and* $i = 1, 2$, $i \neq j$ *then for all risk-averse investors and for* $i = 1, 2$, $i \neq j$ $\alpha_i^* \geq 0$ *if and only if* $E\left(R_i\right) \geq r_0$, *In this case, the position to take on* R_j *(long vs. short) will depend on the covariance between* R_1 *and* R_2.

Proof: See the Appendix.

Note that since a nil covariance is equivalent to independence in the class of quadrant dependent distributions (Lehmann, 1966), the position on R_i, $i = 1, 2$ will also depend on its risk premium, if the covariance between R_1 and R_2 is nil. In the next section, we discuss examples of quadrant dependent distributions (for other examples see Lehmann, 1959; Tong, 1980).

3. Examples

Example 1. A quadrant dependent distribution. Let $R_1 = a + dR_2 + U$, where R_2 and *U* are independent. Then (R_1, R_2) is positively or negatively quadrant dependent as the parameter $d \geq$ or ≤ 0. In particular, if R_2 and U are normally distributed, R_1 and R_2 become the components of a bivariate normal distribution having the same sign for correlation as d.

The second example concerns the set of distributions that allow for two-fund separation as defined by Ross (1978). This set is related to the set of quadrant dependent distributions since many distributions generating separation are included in the set of quadrant dependent distributions. Examples are the normal and the *t*-distribution.

An interesting question is the following: Can quadrant dependence only be satisfied by a two-fund separating distribution? The next example addresses this question and shows that quadrant dependence does not imply two-fund separation.

Example 2. We consider a simple case with three states of the world: The corresponding returns are −3, 1 and 3 for the second risky asset and −2, 1 and 2 for the first risky asset. We assume a zero risk-free interest rate. Table 1 gives the joint density of the returns with perfect correlation (the result can be obtained without this assumption). The proof of quadrant dependence for this distribution is straightforward.

(Table 1 about here)

We consider two risk-averse investors with preferences given respectively by $u_1(W) = W - \frac{1}{4}W^2$ (with $1 - \frac{1}{2}W > 0$ for all W), and

$$u_2(W) = \begin{cases} W-1 \text{ if } W \le 1 \\ \frac{1}{3}(W-1) \text{ if } 1 \le W \le 2. \\ \frac{1}{3} \text{ if } W \ge 2. \end{cases} \quad (7)$$

Note that investor u_2 is not in the Cass-and-Stiglitz (1970) family of separating functions. The optimal investments in the two risky assets for investor u_1 are $(8/3, -5/3)$. The optimal choice for investor u_2 is given by the semi-line $2\alpha_1^* + 3\alpha_2^* = 0$ and $\alpha_1^* + \alpha_2^* \ge 1$; this does not include the optimal choice for investor u_1, as illustrated in Figure 1 (see Example 2 in the Appendix for details). Two-fund separation is then not allowed by the quadrant dependent distribution provided in Table 1.

(Figure 1 about here)

Other examples (Lehmann, 1966; Tong, 1980). The Cauchy distribution (given that $r_2 \in [0,1]$) is a positive quadrant dependent distribution. The main difference between the normal distribution and the Cauchy distribution is the longer and flatter tails of the latter. Other examples of a negative quadrant dependent distribution are the bivariate Dirichlet and the bivariate hypergeometric. The Dirichlet extends the beta distribution to multivariate distributions.

4. Conclusion

We have proposed the concept of quadrant dependence (Lehmann, 1966) to analyze portfolio choice. This concept describes how two random variables behave together when they are simultaneously small or large. By assuming that the returns on risky assets are quadrant dependent, we were able to decompose the demand for one risky asset into an investment part based on the risk premium offered by the asset and a hedging part used against the fluctuations in the return on the other risky asset. Our characterization of the optimal portfolio was done for all risk-averse investors. Quadrant dependence was shown to be related to two-fund separating distributions (Ross, 1978). These results open up the search for broader asset-pricing models that can free stochastic dependence from its connection with linear correlation.

Several extensions of our article are possible. For example, we may look at orthant dependent distributions. Orthant dependence generalizes the bivariate notion of quadrant dependence to higher dimensions. Intuitively, as for PQD, $R_1, R_2, ..., R_n$ are positive orthant dependent if they are more likely to have simultaneously large values as they would be were they independent. A natural and significant extension to our framework would be to verify whether orthant dependence can result in a similar decomposition between the investment component and the hedging component for portfolios with more than two risky assets. Affiliated random variables (Milgrom and Weber, 1982) can also be used for such extension.

Denuit and Scaillet (2004) provided two-test procedures for positive quadrant dependence which are closely related to those proposed by Davidson and Duclos (2000). These procedures did not reject the positive quadrant dependence among data for US and Danish insurance claims. Mimouni (2002) applied the two-test procedures to data on financial assets and found that positive quadrant dependence was not rejected. Further developments of these tests, for portfolios containing many stocks and derivatives, are open for future research. A last extension would be to consider how the variation of different measures of association could affect the results. Covariance is a measure of association. Another one is the quadrant measure of association (q) discussed by Blomqvist (1950). Lehmann (1966) showed that if (R_1, R_2) is positively quadrant dependent, then both $\mathrm{Cov}(R_1, R_2)$ and q are non-negative.

Up to now, the analysis was limited to two random variables which may reduce the applicability of the model. As a referee suggested, the model could be very useful to study the asset allocation between a stock index, a bond index, and money (or a risk free asset). Recently, Elton and Gruber (2000) used that framework to analyze the asset allocation puzzle. It would be interesting to see how the notion of quadrant dependence could add more insights to solve the puzzle.

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Table 1

Joint Density Function of Example 2

This table presents the joint density function of a quadrant dependent distribution. Quadrant dependence describes how two random variables behave together when they are simultaneously small or large. Here we observe that the two assets are positively quadrant dependent. Notice that $E(R_1) = 0.83$, $E(R_2) = 1$, $\sigma_{11} = 1.81$, $\sigma_{22} = 4$, and $\sigma_{12} = 2.7$.

	r_2		
r_1	-3	1	3
-2	1/6	0	0
1	0	1/2	0
2	0	0	1/3

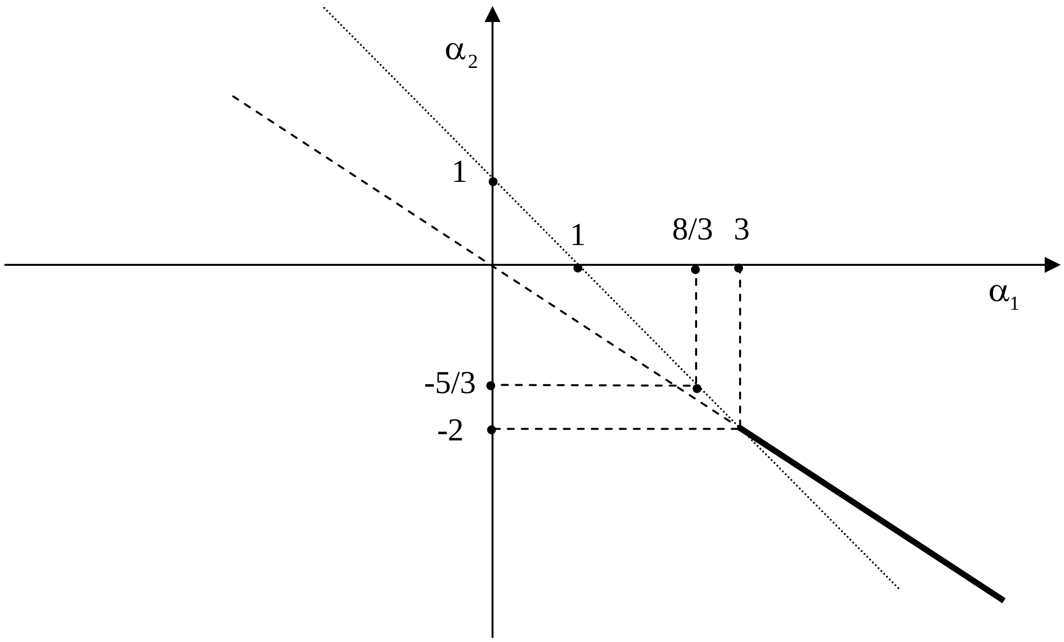

Figure 1. Optional solutions to Example 2. This figure depicts the optimal solution of investor u_1 at (8/3, -5/3) and that of investor u_2 corresponding to the semi-line $2\alpha_1^* + 3\alpha_2^* = 0$ and $\alpha_1^* + \alpha_2^* \geq 1$ when the data are from Table 1. The joint distribution of this example does not yield a separating solution since the point (8/3, -5/3) is not on the semi-line.

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Appendix

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Derivation of α_1^* and α_2^* in the mean-variance model

$$E\left(u\left(W\right)\right) = 1 + r_0 + \alpha_1 m_1 + \alpha_2 m_2 - \frac{b}{2} E\left(\left(\left(1+r_0\right) + \alpha_1\left(R_1 - r_0\right) + \alpha_2\left(R_2 - r_0\right)\right)^2\right).$$

The FOC with respect to α_1 is

$$\begin{aligned} & m_1 - bE\left(\left(R_1 - r_0\right)\left(1 + r_0 + \alpha_1\left(R_1 - r_0\right) + \alpha_2\left(R_2 - r_0\right)\right)\right) \\ &= m_1 - b\left(1+r_0\right)m_1 - b\alpha_1 E\left(R_1 - r_0\right)^2 - b\alpha_2 E\left(\left(R_1 - r_0\right)\left(R_2 - r_0\right)\right) = 0. \end{aligned} \tag{A0}$$

Moreover,

$$\begin{aligned} E\left(R_1 - r_0\right)^2 &= E\left(\left(\left(R_1 - E\left(R_1\right)\right) + E\left(R_1\right) - r_0\right)^2\right) \\ &= E\left(\left(R_1 - E\left(R_1\right)\right)^2\right) + 2E\left(R_1 - E\left(R_1\right)\right)m_1 + \left(E\left(R_1\right) - r_0\right)^2 \\ &= \sigma_{11} + m_1^2 \end{aligned}$$

$$\begin{aligned} E\left(\left(R_1 - r_0\right)\left(R_2 - r_0\right)\right) &= E\left(\left(R_1 - E\left(R_1\right) + E\left(R_1\right) - r_0\right)\left(R_2 - r_0\right)\right) \\ &= E\left(\left(R_1 - E\left(R_1\right)\right)\left(R_2 - r_0\right)\right) + m_1 m_2 \\ &= \sigma_{12} + m_1 m_2. \end{aligned}$$

The FOC with respect to α_1 (A0) can now be written as

$$m_1 - b\left(1+r_0\right)m_1 - b\alpha_1\left(\sigma_{11} + m_1^2\right) - b\alpha_2\left(\sigma_{12} + m_1 m_2\right) = 0.$$

By symmetry, the FOC with respect to α_2 can be written as

$$m_2 - b(1+r_0)m_2 - b\alpha_2\left(\sigma_{22} + m_2^2\right) - b\alpha_1\left(\sigma_{12} + m_1 m_2\right) = 0.$$

The optimal portfolio is the solution to the system

$$\begin{cases} \alpha_1\left(\sigma_{11} + m_1^2\right) + \alpha_2\left(\sigma_{12} + m_1 m_2\right) = \dfrac{1}{b}\left(m_1 - b(1+r_0)m_1\right) \\ \alpha_1\left(\sigma_{12} + m_1 m_2\right) + \alpha_2\left(\sigma_{22} + m_2^2\right) = \dfrac{1}{b}\left(m_2 - b(1+r_0)m_2\right) \end{cases}$$

or as a matrix format

$$A\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = B$$

with

$$A = \begin{pmatrix} \sigma_{11} + m_1^2 & \sigma_{12} + m_1 m_2 \\ \sigma_{12} + m_1 m_2 & \sigma_{22} + m_2^2 \end{pmatrix} \qquad B = \begin{pmatrix} \dfrac{1}{b}\left(m_1 - b(1+r_0)m_1\right) \\ \dfrac{1}{b}\left(m_2 - b(1+r_0)m_2\right) \end{pmatrix}.$$

The solution to this system is

$$\begin{pmatrix} \alpha_1^* \\ \alpha_2^* \end{pmatrix} = \frac{1}{\Delta}\begin{pmatrix} -\sigma_{22} - m_2^2 & \sigma_{12} + m_1 m_2 \\ \sigma_{12} + m_1 m_2 & -\sigma_{11} - m_1 m_2 \end{pmatrix} \times B$$

or

$$\alpha_1^* = \frac{1-b(1+r_0)}{b}\frac{m_1\sigma_{22} - m_2\sigma_{12}}{\Delta}$$

$$\alpha_2^* = \frac{1-b(1+r_0)}{b}\frac{m_2\sigma_{11} - m_1\sigma_{12}}{\Delta}.$$

Finally, the calculation of Δ, the determinant of A, is as follows:

$$
\begin{aligned}
\Delta &= \begin{vmatrix} \sigma_{11} + m_1^2 & \sigma_{12} + m_1 m_2 \\ \sigma_{12} + m_1 m_2 & \sigma_{22} + m_2^2 \end{vmatrix} \\
&= \left(\sigma_{11} + m_1^2\right)\left(\sigma_{22} + m_2^2\right) - \left(\sigma_{12} + m_1 m_2\right)^2 \\
&= \sigma_{11}\sigma_{22} + \sigma_{11} m_2^2 + \sigma_{22} m_1^2 + m_1^2 m_2^2 - \sigma_{12}^2 - 2 m_1 m_2 \sigma_{12} - m_1^2 m_2^2 \\
&= \sigma_{11} m_2^2 + \sigma_{22} m_2^2 - 2 m_1 m_2 \sigma_{12} + \sigma_{11}\sigma_{22} - \sigma_{12}^2 .
\end{aligned}
$$

Proof of Proposition 1: Since the problem is symmetric we only prove the decomposition for α_2^*. Also, for the presentation, we suppose $\alpha_1^* \geq 0$ and we restrict our analysis to PQD. The proof for negative quadrant dependence and $\alpha_1^* \leq 0$ is similar.

The first-order condition with respect to α_2 can be written as

$$
\begin{aligned}
\frac{\partial}{\partial \alpha_2} E\left(u\left(W\left(\alpha_1^*, \alpha_2^*\right)\right)\right) &= \int_{\underline{r}_2}^{\overline{r}_2} \int_{\underline{r}_1}^{\overline{r}_1} \left(r_2 - E\left(R_2\right)\right) u'\left(W\left(\alpha_1^*, \alpha_2^*\right)\right) dF\left(r_1, r_2\right) \\
&\quad + m_2 \int_{\underline{r}_2}^{\overline{r}_2} \int_{\underline{r}_1}^{\overline{r}_1} u'\left(W\left(\alpha_1^*, \alpha_2^*\right)\right) dF\left(r_1, r_2\right) = 0.
\end{aligned}
\tag{A1}
$$

Let α_{2h}^* be the solution to

$$
E\left(\left(R_2 - E\left(R_2\right)\right) u'\left(W\left(\alpha_1^*, \alpha_{2h}\right)\right)\right) = \mathrm{Cov}\left(R_2, u'\left(W\left(\alpha_1^*, \alpha_{2h}\right)\right)\right) = 0 . \tag{A2}
$$

The proof for part *b)* of Proposition 1 is done in two folds. First we prove that (A2) cannot have a positive solution, if it has any; and second, we prove the existence of a solution to (A2). We will get back to this but for now suppose that α_{2h}^* exists and let us present the proof of part *a)* of the proposition.

Part a):

By (A2), (A1) can be rewritten as

$$\frac{\partial}{\partial \alpha_2} E\left(u\left(W\left(\alpha_1^*, \alpha_{2h}^*\right)\right)\right) = m_2 \int_{\underline{r}_2}^{\overline{r}_2} \int_{\underline{r}_1}^{\overline{r}_1} u'\left(W\left(\alpha_1^*, \alpha_{2h}^*\right)\right) dF\left(r_1, r_2\right). \qquad \text{(A3)}$$

It follows from the concavity of the objective function that $\alpha_2^* \geq \alpha_{2h}^*$ if and only if $m_2 \geq 0$, or that $\alpha_2^* - \alpha_{2h}^*$ has the same sign as m_2. Defining $\alpha_{2m}^* = \alpha_2^* - \alpha_{2h}^*$ ends the proof of part *a).*

Part b):

We now prove part *b)* of the proposition. As we already said we need to prove that (A2) cannot have a positive solution, if it has any; and second, we prove the existence of a solution to (A2).

Let us prove that (A2) cannot have a positive solution. We use the next property (P) that follows from positive quadrant dependence:

$$\mathrm{Cov}\left(f(R_1), g(R_2)\right) \geq 0 \text{ for all nondecreasing functions } f \text{ and } g \text{ (see Tong, 1980).} \qquad \text{(P)}$$

For $\alpha_{2h} > 0$ and since u' is decreasing we have:

- For $\underline{r}_2 \leq r_2 \leq E\left(R_2\right)$:

$$r_2 - E\left(R_2\right) \leq 0$$

and

$$u'\left(1 + r_0 + \alpha_1^*\left(r_1 - r_0\right) + \alpha_{2h}\left(r_2 - r_0\right)\right) > u'\left(1 + r_0 + \alpha_1^*\left(r_1 - r_0\right) + \alpha_{2h}\left(E\left(R_2\right) - r_0\right)\right). \qquad \text{(A4)}$$

It follows that:

$$\left(r_2 - E\left(R_2\right)\right)u'\left(1+r_0+\alpha_1^*\left(r_1-r_0\right)+\alpha_{2h}\left(r_2-r_0\right)\right)$$
$$\leq\left(r_2 - E\left(R_2\right)\right)u'\left(1+r_0+\alpha_1^*\left(r_1-r_0\right)+\alpha_{2h}\left(E\left(R_2\right)-r_0\right)\right). \tag{A5}$$

- For $E\left(R_2\right)\leq r_2 \leq \overline{r}_2$:

$$r_2 - E\left(R_2\right)\geq 0$$

and

$$u'\left(1+r_0+\alpha_1^*\left(r_1-r_0\right)+\alpha_{2h}\left(r_2-r_0\right)\right) < u'\left(1+r_0+\alpha_1^*\left(r_1-r_0\right)+\alpha_{2h}\left(E\left(R_2\right)-r_0\right)\right). \tag{A6}$$

It follows that:

$$\left(r_2 - E\left(R_2\right)\right)u'\left(1+r_0+\alpha_1^*\left(r_1-r_0\right)+\alpha_{2h}\left(r_2-r_0\right)\right)$$
$$\leq\left(r_2 - E\left(R_2\right)\right)u'\left(1+r_0+\alpha_1^*\left(r_1-r_0\right)+\alpha_{2h}\left(E\left(R_2\right)-r_0\right)\right).$$

(A1) can be written as:

$$\int_{\underline{r}_2}^{\overline{r}_2}\int_{\underline{r}_1}^{\overline{r}_1}\left(r_2 - E\left(R_2\right)\right)u'\left(W\left(\alpha_1^*,\alpha_{2h}\right)\right)\,dF\left(r_1,r_2\right) = \int_{\underline{r}_2}^{E(R_2)}\int_{\underline{r}_1}^{\overline{r}_1}\left(r_2 - E\left(R_2\right)\right)u'\left(W\left(\alpha_1^*,\alpha_{2h}\right)\right)dF\left(r_1,r_2\right)$$
$$+\int_{E(R_2)}^{\overline{r}_2}\int_{\underline{r}_1}^{\overline{r}_1}\left(r_2 - E\left(R_2\right)\right)u'\left(W\left(\alpha_1^*,\alpha_{2h}\right)\right)dF\left(r_1,r_2\right). \tag{A7}$$

By integrating over $\left[\underline{r}_2, E\left(R_2\right)\right]$ and $\left[E\left(R_2\right),\overline{r}_2\right]$, we obtain that, for $\alpha_{2h} > 0$, (A7) is lower than:

$$\int_{\underline{r}_2}^{E(R_2)}\int_{\underline{r}_1}^{\overline{r}_1}\left(r_2 - E\left(R_2\right)\right)u'\left(1+r_0+\alpha_1^*\left(r_1-r_0\right)+\alpha_{2h}\left(E\left(R_2\right)-r_0\right)\right)dF\left(r_1,r_2\right)$$

$$+\int_{E(R_2)}^{\overline{r}_2}\int_{\underline{r}_1}^{\overline{r}_1}\left(r_2-E\left(R_2\right)\right)u'\left(1+r_0+\alpha_1^*(r_1-r_0)+\alpha_{2h}\left(E(R_2)-r_0\right)\right)dF\left(r_1,r_2\right).$$

The sum of the last two integrals gives:

$$\int_{\underline{r}_2}^{\overline{r}_2}\int_{\underline{r}_1}^{\overline{r}_1}\left(r_2-E\left(R_2\right)\right)u'\left(1+r_0+\alpha_1^*\left(r_1-r_0\right)+\alpha_{2h}\left(E\left(R_2\right)-r_0\right)\right)dF\left(r_1,r_2\right), \tag{A8}$$

which is equal to:

$$Cov\left(R_2,u'\left(1+r_0+\alpha_1^*\left(R_1-r_0\right)+\alpha_{2h}\left(E\left(R_2\right)-r_0\right)\right)\right)\leq 0. \tag{A9}$$

The last inequality follows from property (P) where

$$f\left(r_1\right)=-u'\left(1+r_0+\alpha_1^*\left(r_1-r_0\right)+\alpha_{2h}\left(E\left(R_2\right)-r_0\right)\right)$$

and $g(r_2)=r_2$.

The solution to (A2) is then certainly negative since for $\alpha_{2h}>0$,

$$E\left(\left(R_2-E\left(R_2\right)\right)u'\left(W\left(\alpha_1^*,\alpha_{2h}\right)\right)\right) \tag{A10}$$

is strictly negative.

To prove the existence of α_{2h}^*, we make use of the theorem of the intermediate value. By the continuity of $E\left(\left(R_2-E\left(R_2\right)\right)u'\left(W\left(\alpha_1^*,\alpha_{2h}\right)\right)\right)$ in α_{2h}, we will be done if we prove that there exists a $\overline{\alpha}_{2h}$ where $E\left(\left(R_2-E\left(R_2\right)\right)u'\left(W\left(\alpha_1^*,\overline{\alpha}_{2h}\right)\right)\right)$ is positive (we already know from (A10) that $E\left(\left(R_2-E\left(R_2\right)\right)u'\left(W\left(\alpha_1^*,\alpha_{2h}\right)\right)\right)$ takes negative values).

After integration by parts, (A2) simplifies to:

$$-\int_{\underline{r}_2}^{\overline{r}_2} \theta(r_2) K^{'}(r_2) dr_2,$$

where

$$\theta(r_2) = \int_{\underline{r}_2}^{r_2} (t - E(R_2)) dG(t),\ K(r_2) = \int_{\underline{r}_1}^{\overline{r}_1} u'\left(1 + \alpha_1^*(r_1 - r_0) + \alpha_{2h}(r_2 - r_0)\right) dF(r_1 |\ r_2)$$

and

$$dF(r_1, r_2) = dF(r_1 |\ r_2) dG(r_2).$$

$K'(r_2)$ is analyzed in more details below. Since $\theta(\cdot)$ is negative, to complete the proof we need to show that $K'(r_2)$ is positive for a given value of α_{2h}.

Let *n* be a positive integer and replace α_{2h} by $-n\alpha_1^*$ in $K(r_2)$. $K'(r_2)$ simplifies to

$$K'(r_2) = -\alpha_1^* \int_{\underline{r}_1}^{\overline{r}_1} u''\left(1 + \alpha_1^*(r_1 - r_0) - n\alpha_1^*(r_2 - r_0)\right)\left[n \frac{\partial}{\partial r_1} F(r_1 |\ r_2) + \frac{\partial}{\partial r_2} F(r_1 |\ r_2)\right] dr_1. \qquad \text{(A11)}$$

By continuity, and since $\left[\underline{r}_1, \overline{r}_1\right]$ and $\left[\underline{r}_2, \overline{r}_2\right]$ are compact and $\frac{\partial}{\partial r_1} F(r_1 |\ r_2)$ is positive, there exists at least one $\overline{n}$ where $\overline{n} \frac{\partial}{\partial r_1} F(r_1 |\ r_2) + \frac{\partial}{\partial r_2} F(r_1 |\ r_2) \geq 0$ for all r_1, r_2. As a result, for $\alpha_{2h} = -\overline{n}\alpha_1^*$, $K'(r_2)$ is positive for all $r_2 \in \left[\underline{r}_2, \overline{r}_2\right]$, and hence

$$E\left(\left(R_2 - E(R_2)\right) u'\left(1 + r_0 + \alpha_1^*(r_1 - r_0) - \alpha_1^* \overline{n}(r_2 - r_0)\right)\right) \geq 0.$$

To complete the proof, we analyze in more detail the expression $K'(r_2)$. By definition

$$K(r_2) = \int_{\underline{r}_1}^{\overline{r}_1} u'\left(1 + \alpha_1^*(r_1 - r_0) + \alpha_{2h}(r_2 - r_0)\right) dF(r_1 | r_2),$$

and its derivative is

$$K'(r_2) = \alpha_{2h}\int_{\underline{r}_1}^{\bar{r}_1} u''\left(1+\alpha_1^*(r_1-r_0)+\alpha_{2h}(r_2-r_0)\right)dF(r_1\,|\,r_2)$$
$$+\int_{\underline{r}_1}^{\bar{r}_1} u'\left(1+\alpha_1^*(r_1-r_0)+\alpha_{2h}(r_2-r_0)\right)\frac{\partial}{\partial r_2}dF(r_1\,|\,r_2).$$

The second term in the last expression can be written, after integration by parts, as

$$\int_{\underline{r}_1}^{\bar{r}_1} u'\left(1+\alpha_1^*(r_1-r_0)+\alpha_{2h}(r_2-r_0)\right)\frac{\partial}{\partial r_2}dF(r_1\,|\,r_2) = u'\left(1+\alpha_1^*(\bar{r}_1-r_0)+\alpha_{2h}(r_2-r_0)\right)\frac{\partial}{\partial r_2}F(\bar{r}_1\,|\,r_2)$$
$$-u'\left(1+\alpha_1^*(\underline{r}_1-r_0)+\alpha_{2h}(r_2-r_0)\right)\frac{\partial}{\partial r_2}F(\underline{r}_1\,|\,r_2)- \alpha_1^*\int_{\underline{r}_1}^{\bar{r}_1} u''\left(1+\alpha_1^*(r_1-r_0)+\alpha_{2h}(r_2-r_0)\right)\frac{\partial}{\partial r_2}F(r_1\,|\,r_2)dr_1$$
$$= -\alpha_1^*\int_{\underline{r}_1}^{\bar{r}_1} u''\left(1+\alpha_1^*(r_1-r_0)+\alpha_{2h}(r_2-r_0)\right)\frac{\partial}{\partial r_2}F(r_1\,|\,r_2)dr_1. \qquad \text{(A12)}$$

The last inequality follows from

$$\frac{\partial}{\partial r_2}F(\bar{r}_1\,|\,r_2) = \frac{\partial}{\partial r_2}F(\underline{r}_1\,|\,r_2) = 0.$$

So $K'(r_2)$ can be written as:

$$K'(r_2) = \alpha_{2h}\int_{\underline{r}_1}^{\bar{r}_1} u''\left(1+\alpha_1^*(r_1-r_0)+\alpha_{2h}(r_2-r_0)\right)dF(r_1\,|\,r_2) - \alpha_1^*\int_{\underline{r}_1}^{\bar{r}_1} u''\left(1+\alpha_1^*(r_1-r_0)+\alpha_{2h}(r_2-r_0)\right)\frac{\partial}{\partial r_2}F(r_1\,|\,r_2)dr_1. \qquad \text{(A13)}$$

We can now write $K'(r_2)$ for $\alpha_{2h} = -n\alpha_1^*$ as:

$$= -\alpha_1^*\int_{\underline{r}_1}^{\bar{r}_1} u''\left(1+\alpha_1^*(r_1-r_0)-n\alpha_1^*(r_2-r_0)\right)\left(ndF(r_1\,|\,r_2)+\frac{\partial}{\partial r_2}dF(r_1\,|\,r_2)dr_1\right)$$

$$= -\alpha_1^*\int_{\underline{r}_1}^{\bar{r}_1} u''\left(1+\alpha_1^*(r_1-r_0)-n\alpha_1^*(r_2-r_0)\right)\left(n\frac{\partial}{\partial r_1}F(r_1\,|\,r_2)+\frac{\partial}{\partial r_2}F(r_1\,|\,r_2)\right)dr_1$$

which is (A6). This ends the proof of the existence of α_{2h}^{*} and the proof of **Proposition 1**.

Q.E.D.

Proof of Proposition 2: By symmetry we only prove the result for α_2^{*}. We write the joint distribution of (R_1, R_2) as $dF(r_1 \mid r_2)dG(r_2)$. We know that if $\mathrm{Cov}(R_1, R_2) = 0$ then $\alpha_{2h}^{*} = 0$. It remains to show that if $\alpha_{2h}^{*} = 0$ then the two random variables have a nil covariance. Integrating by parts the left-hand-side term in (A2) yields

$$-\int_{\underline{r}_2}^{\overline{r}_2}\left(\int_{\underline{r}_2}^{r_2}\left(t - E(R_2)dG(t)\right)\right)\left(\int_{\underline{r}_1}^{\overline{r}_1} u'\left(W\left(\alpha_1^{*}, 0\right)\right)\frac{\partial}{\partial r_2} dF\left(r_1 \mid r_2\right)\right)dr_2 = 0 . \qquad \text{(A14)}$$

Under our assumption of quadrant dependence and since $\int_{\underline{r}_2}^{r_2}\left(t - E(R_2)\right)dG(t) \le 0$ for all r_2, in order for equality in (A14) to hold under the assumption of quadrant dependance, we need to have

$$\frac{\partial}{\partial r_2} F\left(r_1 \mid r_2\right) = 0 \text{ for all } r_2 ,$$

which means that R_1 and R_2 have a nil covariance.

Part b) of the proposition follows from Proposition 1.

Q.E.D.

Proof of Proposition 3: Proving the first part of Proposition 3 is equivalent to proving that

$$\mathrm{Sign}\left(\alpha_1^{*}\right) = \mathrm{Sign}\left(m_1\right). \qquad \text{(A15)}$$

Since the agent is risk averse he will always prefer the certainty equivalent to a gamble with the same expected return. In fact, with Jensen’s inequality, one has

$$\begin{aligned} E\left(u\left(W\left(\alpha_1^*,0\right)\right)\right) &\le u\left(E\left(W\left(\alpha_1^*,0\right)\right)\right) \\ &= u\left(1+r_0+m_1\alpha_1^*\right). \end{aligned} \tag{A16}$$

If α_1^* and m_1 have opposite signs then $m_1\alpha_1^* < 0$ and hence

$$E\left(u\left(W\left(\alpha_1^*,0\right)\right)\right) < u\left(1+r_0\right). \tag{A17}$$

The latter inequality contradicts the optimality of $\left(\alpha_1^*,0\right)$ since $(0,0)$ is a better investment strategy. Consequently, $m_1 \ge 0$ is a necessary and sufficient condition for $\alpha_1^* \ge 0$. In addition, from Proposition 2, and since $m_2 = 0$, we know that $\alpha_{2m}^* = 0$. The optimal position to take on R_2 is then given by part b) of Proposition 1.

Q.E.D.

Example 2: Since r_0 is normalized to 0, the random end-of-period wealth is $W\left(\alpha_1,\alpha_2\right) = 1+\alpha_1 R_1+\alpha_2 R_2$.

From Table 1, the expected utility function of the second agent for an investment $\left(\alpha_1,\alpha_2\right)$ is

$$E\left(u_2\right) = \frac{1}{6}u_2\left(1-2\alpha_1-3\alpha_2\right)+\frac{1}{2}u_2\left(1+\alpha_1+\alpha_2\right)+\frac{1}{3}u_2\left(1+2\alpha_1+3\alpha_2\right), \tag{A18}$$

where

$$u_2\left(1-2\alpha_1-3\alpha_2\right) = \begin{cases} -2\alpha_1-3\alpha_2 \text{ if } 2\alpha_1+3\alpha_2 \ge 0 \\ \frac{1}{3}\left(-2\alpha_1-3\alpha_2\right) \text{ if } -1 \le 2\alpha_1+3\alpha_2 \le 0 \\ \frac{1}{3} \text{ if } 2\alpha_1+3\alpha_2 \le -1. \end{cases} \tag{A19}$$

$$u_2\left(1+\alpha_1+\alpha_2\right)=\begin{cases}\alpha_1+\alpha_2 \text{ if } \alpha_1+\alpha_2\le 0\\ \frac{1}{3}\left(\alpha_1+\alpha_2\right) \text{ if } 0\le\alpha_1+\alpha_2\le 1\\ \frac{1}{3} \text{ if } \alpha_1+\alpha_2\ge 1.\end{cases} \tag{A20}$$

$$u_2\left(1+2\alpha_1+3\alpha_2\right)=\begin{cases}2\alpha_1+3\alpha_2 \text{ if } 2\alpha_1+3\alpha_2\le 0\\ \frac{1}{3}\left(2\alpha_1+3\alpha_2\right) \text{ if } 0\le 2\alpha_1+3\alpha_2\le 1\\ \frac{1}{3} \text{ if } 2\alpha_1+3\alpha_2\ge 1.\end{cases} \tag{A21}$$

We have 12 different scenarios for $\left(\alpha_1,\alpha_2\right)$ that we need to discuss in order to solve for the optimal portfolio. As can be seen from $u_2\left(1+\alpha_1+\alpha_2\right)$, the investor is always better off with $\alpha_1+\alpha_2\ge 1$. This reduces the number of cases for analysis to 4.

We look at local maximum for each of the 4 possible cases.

1. $2\alpha_1+3\alpha_2\le -1,\ \alpha_1+\alpha_2\ge 1$

$$E\left(u_2\right)=\frac{1}{6}*\frac{1}{3}+\frac{1}{2}*\frac{1}{3}+\frac{1}{3}\left(2\alpha_1+3\alpha_2\right).$$

The expected utility is clearly maximized at $2\alpha_1+3\alpha_2=-1$, and the maximum utility in this semi-plan is $E\left(u_2\right)=\frac{1}{6}*\frac{1}{3}+\frac{1}{2}*\frac{1}{3}-\frac{1}{3}=-\frac{1}{9}$.

2. $-1\le 2\alpha_1+3\alpha_2\le 0,\ \alpha_1+\alpha_2\ge 1$

$$\begin{aligned}E\left(u_2\right)&=\frac{1}{3}*\frac{1}{6}\left(-2\alpha_1-3\alpha_2\right)+\frac{1}{2}*\frac{1}{3}+\frac{1}{3}\left(2\alpha_1+3\alpha_2\right)\\&=-\frac{5}{18}\left(2\alpha_1+3\alpha_2\right)+\frac{1}{6}.\end{aligned}$$

It follows that the expected utility is maximized at $2\alpha_1 + 3\alpha_2 = 0,$ and the maximum utility achieved in the semi-plan is $E(u_2) = -\frac{5}{18} * 0 + \frac{1}{6} = \frac{1}{6}$.

3. $0 \le 2\alpha_1 + 3\alpha_2 \le 1,\ \alpha_1 + \alpha_2 \ge 1$

$$\begin{aligned} E(u_2) &= \frac{1}{6}(-2\alpha_1 - 3\alpha_2) + \frac{1}{2} * \frac{1}{3} + \frac{1}{3} * \frac{1}{3}(2\alpha_1 + 3\alpha_2) \\ &= -\frac{1}{18}(2\alpha_1 + 3\alpha_2) + \frac{1}{6}. \end{aligned}$$

the maximum is clearly achieved at $2\alpha_1 + 3\alpha_2 = 0,$ and the maximum utility achieved in the semi-plan is $E(u_2) = -\frac{1}{18} * 0 + \frac{1}{6} = \frac{1}{6}$.

4. $1 \le 2\alpha_1 + 3\alpha_2,\ \alpha_1 + \alpha_2 \ge 1$

$$\begin{aligned} E(u_2) &= \frac{1}{6}(-2\alpha_1 - 3\alpha_2) + \frac{1}{2} * \frac{1}{3} + \frac{1}{3} * \frac{1}{3} \\ &= -\frac{1}{6}(2\alpha_1 + 3\alpha_2) + \frac{1}{6} + \frac{1}{9}. \end{aligned}$$

Since $-2\alpha_1 - 3\alpha_2 \le -1$, the maximum is obtained at $2\alpha_1 + 3\alpha_2 = 1$, and the maximum utility in this semi-plan is $E(u_2) = -\frac{1}{6} + \frac{1}{6} + \frac{1}{9} = \frac{1}{9}$.

The global maximum is then the set $\{(\alpha_1, \alpha_2) / 2\alpha_1 + 3\alpha_2 = 0, \alpha_1 + \alpha_2 \ge 1\},$ in which the maximum utility level achieved is $\frac{1}{6}$.