Exotic Options Pricing under Stochastic Volatility

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Abstract

This paper proposes an analytical approximation to price exotic options within a stochastic volatility framework. Assuming a general mean reverting process for the underlying asset and a square-root process for the volatility, we derive an approximation for option prices using a Taylor expansion around two average defined volatilities. The moments of the average volatilities are computed analytically at any order using a Frobenius series solution to some ordinary differential equation. Pricing some exotics such as barrier and digital barrier options, the approximation is found to be very efficient and convergent even at low Taylor expansion order.

Keywords: Option pricing; Exotic options; Digital barrier options; Mean reversion; Stochastic volatility; Frobenius series.

JEL Classification: G13, C63

Résumé

Cet article propose une approximation analytique pour évaluer les options exotiques dans un cadre de volatilité stochastique. En considérant un processus avec retour à la moyenne pour l'actif sous-jacent et un processus racine-carrée pour la volatilité, on dérive une approximation pour les options en utilisant un développement de Taylor autour de deux volatilités « moyennes » qui seront définies. Les moments des volatilités moyennes sont calculés analytiquement en utilisant une solution en séries de Frobenius d'une certaine équation différentielle ordinaire. En évaluant certaines options exotiques comme les options barrières et les options barrières digitales, on montre que l'approximation converge rapidement et qu'elle est très précise.

Mots-clés: Évaluation d'options; Options exotiques; Options barrières digitales; Retour à la moyenne; Volatilité stochastique; Séries de Frobenius.

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Introduction

Several papers propose pricing formulas for plain vanilla options on stocks within different stochastic volatility frameworks. Heston (1993) is the first one who proposes a closed-form price for a standard European call when using a square-root volatility process by inverting the characteristic function seen as a Fourier transform. Bakshi, Cao and Chen (1997) propose an empirical performance study of some alternative option pricing models including stochastic volatility and jumps processes by deriving closed-form solutions in the same way as Heston (1993). Schöbel and Zhu (1999) and Zhu (2000) derive, in a very elegant way, a modular pricing method which includes the square-root and the Ornstein-Uhlenbeck volatility processes mixed eventually with jumps. For some volatility models however, no closed-form solutions can be derived and some numerical techniques are used instead. Hull and White (1987) and Sabanis (2002, 2003) obtain an approximate solution using a Taylor series expansion based on the underlying asset's distribution conditional on the average value of the stochastic variance.

Slightly little work was done for pricing exotic derivatives such as path-dependent options in non-constant volatility models. Davydov and Linetsky (2001) derive closed-form solutions, in terms of Bessel and Whittaker functions, for barrier and lookback options under a constant elasticity of variance diffusion model. Henderson and Hobson (2000) price passport options in Hull and White (1987) and Stein and Stein (1991) stochastic volatility frameworks using the series expansion technique. In both cases, very simple closed-form solutions for the central moments of the average stochastic variance are proposed[1]. This is made possible for some volatility processes such as the geometric Brownian motion and the mean reverting diffusion[2]. Unfortunately, for other stochastic volatility processes such as the square-root diffusion, no simple closed-form formulas for the moments of the average variance can be found. In that case, other techniques such as numerical approximations or Monte Carlo simulation may be used to price derivatives. Apel, Winkler and Wystup (2001) propose a finite elements method to price plain vanilla and barrier options under a square-root stochastic volatility model.

Many assets including interest rates, credit spreads [see Longstaff and Schwartz (1995), Tahani (2000), Prigeant et al. (2001) and Jacobs and Li (2003)] and some

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[1] Sabanis (2002, 2003) derives an iterative procedure to compute these moments
[2] Sabanis (2003) calls "mean reverting" the following volatility diffusion $dV_t = (\kappa\theta - \lambda V_t)dt + \sigma V_t dW_t$.

commodities (Schwartz, 1997) are shown to exhibit a mean reversion feature. But, there is little literature on pricing derivatives for this type of underlying assets. Under stochastic volatility models, most of the work is done on plain vanilla derivatives. Clewlow and Strickland (1997) price standard interest rate derivatives under a square-root volatility model using Monte Carlo simulations. Assuming the latter volatility process, Tahani (2004) prices credit spread options, caps, floors and swaps using Gaussian quadrature. Under a constant volatility assumption, Leblanc and Scaillet (1998) propose some path-dependent interest rate options formulas for the affine term structure model.

This paper proposes to price some exotic options on a mean reverting underlying asset in a square-root volatility model using a series expansion around two average defined volatilities. The choice of this power series method is encouraged by the findings of Ball and Roma (1994) about its accuracy and its easy implementation in comparison to other approaches. The key thing of this method is that the price of a contingent claim may be computed as the expectation of the corresponding constant-volatility model's price where the volatility and the spot price are random variables accounting for stochastic variance [see Hull and White (1987) and Romano and Touzi (1997)]. It remains though to derive the central moments of the average variances and use them in the series expansion. But since the closed-form formulas for these moments can only be derived in term of Whittaker functions[3], which are heavy-computational, it is preferable to compute them using a Frobenius series solution[4] which is very accurate, very fast and very easy to implement.

The next section presents the proposed model and introduces the series expansion method. Section II derives a Frobenius series solution to the moments of the average variances. Section III presents valuation formulas for some exotic options. Section IV presents some numerical results on convergence and efficiency. Section V will conclude.

I The proposed model

Following Tahani (2004), we consider the two stochastic differential equations (SDEs) for the state variable and its volatility under a risk-neutral measure Q :

$$dX_u = \left(\mu - \alpha X_u - \gamma V_u\right)dt + \sqrt{V_u}\left(\rho dB_u + \sqrt{1-\rho^2}\,dW_u\right) \qquad (1)$$

$$dV_u = \left(\kappa\theta - \lambda V_u\right)dt + \sigma\sqrt{V_u}\,dB_u \qquad (2)$$

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[3] In fact, we can derive the moments of the average variances in term of derivatives of Whittaker functions w.r.t. the first and the third arguments.
[4] Selby and Strickland (1995) use the same technique to avoid the use of confluent hypergeometric functions in the Fong and Vasicek (1992) bond price formula.

where $\{\exp(X_u), t \le u \le T\}$ is the price process of a primitive asset such as a stock or a credit spread, $\{V_u, t \le u \le T\}$ is the volatility process and ρ is the correlation between the state variable and its volatility. W and B are two independent Brownian motions on a probability space $(\Omega, \mathcal{F}, Q)$ and $\{\mathcal{F}_u, t \le u \le T\}$ is the Q-augmentation of the filtration generated by (W, B). The parameters μ, α, γ, κ, θ, λ and σ are constant.

Pricing theory allows us to write the price $p(X_t, V_t, t)$ of any European contingent claim on X as the expectation, under a risk-neutral measure, of the discounted payoff of the contract in order to get :

$$p(X_t, V_t, t) = E^Q\left(e^{-r(T-t)} H\left(\{X_s, t \le s \le T\}, T\right) \mid \mathcal{F}_t\right) \tag{3}$$

where r is the constant risk-free rate, T the contract maturity, H is the payoff that could depends on the whole path of the state variable $(X_s)_{t \le s \le T}$. In order to develop a series expansion approximation to the contract price, we shall adapt the methodology in Romano and Touzi (1997) to our mean reverting process X. Referring to the details in Appendix 3.A, we can write the solution to the SDE (1) as :

$$X_T = e^{-\alpha(T-t)} X_t + Y_{t,T} + \mu \int_t^T e^{-\alpha(T-s)} ds - \gamma \int_t^T e^{-\alpha(T-s)} V_s ds$$
$$+ \frac{1}{2} \rho^2 \int_t^T e^{-2\alpha(T-s)} V_s ds + \sqrt{1-\rho^2} \int_t^T e^{-\alpha(T-s)} \sqrt{V_s} dW_s \tag{4}$$

where

$$Y_{t,T} = \rho \int_t^T e^{-\alpha(T-s)} \sqrt{V_s} dB_s - \frac{1}{2} \rho^2 \int_t^T e^{-2\alpha(T-s)} V_s ds \tag{5}$$

Defining an effective state variable $\tilde{X}$ and some average variances by :

$$\tilde{X}_t = X_t + e^{\alpha(T-t)} Y_{t,T} \tag{6}$$

and :

$$\begin{cases} \overline{V_1} \int_t^T e^{-\alpha(T-s)} ds = \int_t^T e^{-\alpha(T-s)} V_s ds \\ \overline{V_2} \int_t^T e^{-2\alpha(T-s)} ds = \int_t^T e^{-2\alpha(T-s)} V_s ds \\ \overline{V_3} \int_t^T e^{-\alpha(T-s)} ds = \int_t^T e^{-2\alpha(T-s)} V_s ds \end{cases} \tag{7}$$

leads to conclude that the process $\tilde{X}$ can be viewed as a solution to the following SDE :

$$d\tilde{X}_u = \left(\mu - \gamma \overline{V_1} + \frac{1}{2} \rho^2 \overline{V_3} - \alpha \tilde{X}_u \right) du + \sqrt{(1 - \rho^2) \overline{V_2}} \, dW_u \tag{8}$$

Putting $\alpha = 0$, $\mu = r$ and $\gamma = \dfrac{1}{2}$ leads to the corresponding expressions in Romano and

Touzi (1997) given by :

$$X_T = X_t + Y_{t,T} + r(T - t) - \frac{1}{2}\left(1 - \rho^2\right)\int_t^T V_s \, ds \tag{9}$$

$$+ \sqrt{1 - \rho^2} \int_t^T \sqrt{V_s} \, dW_s$$

$$Y_{t,T} = \rho \int_t^T \sqrt{V_s} \, dB_s - \frac{1}{2} \rho^2 \int_t^T V_s \, ds \tag{10}$$

and

$$\begin{cases} d\tilde{X}_s = \left(r - \frac{1}{2}\left(1 - \rho^2\right)\overline{V} \right) ds + \sqrt{\left(1 - \rho^2\right)\overline{V}} \, dW_s \\[2mm] \overline{V} = \frac{1}{T - t} \int_t^T V_s \, ds \end{cases} \tag{11}$$

The price of the contingent claim given in Equation (3.3) can thus be rewritten as :

$$p(X_t, V_t, t) = E^Q\left(E^Q\left(e^{-r(T-t)} H\left(\{\tilde{X}_s, t \le s \le T\}, T\right) \middle| \mathcal{G}_t \right) \middle| \mathcal{F}_t \right) \tag{12}$$

where $\mathcal{G}_t = \sigma\{W_t, B_u : t \le u \le T\}$ is a new σ-algebra which assumes that the movements of the volatility over the entire life of the contract are known at time t. In their non-zero correlation model, Romano and Touzi (1997) derive the price of a standard European call option in this way as the expectation of the Black and Scholes (1973) price where the underlying asset price is replaced by $\exp(X_t + Y_{t,T})$ and the volatility parameter is replaced

by $\sqrt{\dfrac{1}{T - t} \overline{V}}$. But this is not an explicit formula since one still has to compute the

expectation, which is almost impossible in the non-zero correlation case. Lewis (2000) notices that even in the case of a zero correlation, the pricing formula in Equation (3.3) does not always lead to an analytical solution because the integrated volatility density is difficult to derive in closed-form[5]. Both Hull and White (1987) and Sabanis (2002, 2003)

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[5] In Lewis (2000), page 116.

assume zero correlation and derive an approximation to the European call price. Taking into account this findings and since our aim is to get some explicit pricing formulas for exotic options, we will assume for the remainder of the article that the correlation between the state variable and its volatility is zero[6].

In the zero correlation case, the process $Y_{t,T}$ given in Equation (3.5) is always 0 and the Equations (3.4), (3.7) and (3.8) can be simplified as :

$$X_T = e^{-\alpha(T-t)} X_t + \mu \int_t^T e^{-\alpha(T-s)} ds - \gamma \int_t^T e^{-\alpha(T-s)} V_s ds$$

$$+ \int_t^T e^{-\alpha(T-s)} \sqrt{V_s} dW_s \tag{13}$$

$$dX_s = \left(\mu - \gamma \overline{V_1} - \alpha X_s \right) ds + \sqrt{V_2} dW_s \tag{14}$$

where

$$\overline{V_1} = \frac{\int_t^T e^{-\alpha(T-s)} V_s ds}{\int_t^T e^{-\alpha(T-s)} ds} \quad ; \quad \overline{V_2} = \frac{\int_t^T e^{-2\alpha(T-s)} V_s ds}{\int_t^T e^{-2\alpha(T-s)} ds} \tag{15}$$

The pricing formula in Equation (3.12) can be simplified by first computing a pseudo-price as a function of the two average variances $\left(\overline{V}_1, \overline{V}_2 \right)$:

$$\overline{p}\left(\overline{V}_1, \overline{V}_2 \right) = E^Q \left(e^{-r(T-t)} H\left(\{ X_s, t \le s \le T \}, T \right) \middle| \mathcal{G}_t \right) \tag{16}$$

and then taking the expectation of this pseudo-price conditional on the initial Q-augmented filtration $\{ \mathcal{F}_u, t \le u \le T \}$ to get the price of the contingent claim :

$$p(X_t, V_t, t) = E^Q \left(\overline{p}\left(\overline{V}_1, \overline{V}_2 \right) \middle| \mathcal{F}_t \right) \tag{17}$$

To obtain an approximation for the price in Equation (3.17), $\overline{p}\left(\overline{V}_1, \overline{V}_2 \right)$ is expanded in a Taylor series around the expected values of $\overline{V}_1$ and $\overline{V}_2$. Thus, the expectation on the right-hand side of Equation (3.17) takes the following form :

$$p(X_t, V_t, t) = \sum_{n=0}^{+\infty} \sum_{m=0}^{+\infty} \left\{ \begin{array}{l} \dfrac{1}{n!} \dfrac{1}{m!} E\left[\left(\overline{V}_1 - E(\overline{V}_1) \right)^n \left(\overline{V}_2 - E(\overline{V}_2) \right)^m \right] \\ \times \partial_{n,m} \overline{p}\left(E(\overline{V}_1), E(\overline{V}_2) \right) \end{array} \right\} \tag{18}$$

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where all the expectations are taken under the risk-neutral measure Q conditional on $\mathcal{F}_t$. We must compute the (n,m)-derivatives of the pseudo-price w.r.t. to the average variances and the cross-moments $E\left(\overline{V_1}^n \overline{V_2}^m\right)$ for all orders (n,m)[7]. Since the differentiation of the pseudo-price depends on the contingent claim specifications, it will be done later and we start by deriving the cross-moments given the square-root volatility diffusion in Equation (3.2).

II The Frobenius solution

Define the cross-moment generating function to be given by :

$$g(a,b;t,T,V) \equiv E\left(\exp\left[-a\int_t^T e^{-\alpha(T-s)}V_s ds - b\int_t^T e^{-2\alpha(T-s)}V_s ds\right]\right) \qquad (19)$$

where $E(\cdot)$ denotes $E^Q(\cdot|\mathcal{F}_t)$. Given the expressions of $\overline{V_1}$ and $\overline{V_2}$ in Equation (3.15), we can easily see that the $(n,m)^{\text{th}}$ cross-moment can be written as :

$$E\left(\overline{V_1}^n \overline{V_2}^m\right) = (-1)^{n+m} \times \left(\int_t^T e^{-\alpha(T-s)}ds\right)^{-n} \times \left(\int_t^T e^{-2\alpha(T-s)}ds\right)^{-m} \times \partial_{n,m} g(a,b)\Big|_{(0,0)} \qquad (20)$$

Considering g as a function of (τ,V), Feynman-Kac theorem allows us to write $g(\tau,V)$ as the solution to a partial differential Equation (3.PDE) that takes the following form :

$$\begin{cases} \dfrac{\partial g}{\partial \tau} = \dfrac{1}{2}\sigma^2 V \dfrac{\partial^2 g}{\partial V^2} + (\kappa\theta - \lambda V)\dfrac{\partial g}{\partial V} - \eta(\tau)Vg \\ g(0,V) = 1 \end{cases} \qquad (21)$$

where $\tau = T - t$ and $\eta(\tau) = a\exp(-\alpha\tau) + b\exp(-2\alpha\tau)$. Following Tahani (2004), we assume that g is log-linear in (τ,V) and can be written as :

$$g(\tau,V) = \exp(VD(\tau) + C(\tau)) \qquad (22)$$

where $D(.)$ and $C(.)$ are solutions to the two following ordinary differential equations (ODEs) :

$$\begin{cases} D'(\tau) - \dfrac{1}{2}\sigma^2 D^2(\tau) + \lambda D(\tau) + \eta(\tau) = 0 \\ D(0) = 0 \end{cases} \qquad (23)$$

and

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[7] Hopefully, we won't need to get to very large values of n and m. Sabanis (2002, 2003) and Apel et al (2001) only need the second and find the third order of negligible impact.

$$\begin{cases} C'(\tau) = \kappa\theta \, D(\tau) \\ C(0) = 0 \end{cases} \tag{24}$$

The exact solutions to these ODEs are given by :

$$\begin{cases} D(\tau) = -\dfrac{2}{\sigma^2} \dfrac{U'(\tau)}{U(\tau)} \\[2ex] C(\tau) = -\dfrac{2\kappa\theta}{\sigma^2} \ln\big(U(\tau)\big) \end{cases} \tag{25}$$

where U solves the following linear homogeneous second-order ODE :

$$\begin{cases} U''(\tau) + \lambda U'(\tau) - \dfrac{1}{2}\sigma^2 U(\tau)\eta(\tau) = 0 \\[2ex] U'(0) = 0 \quad , \quad U(0) = 1 \end{cases} \tag{26}$$

Tahani (2004) provides the exact solution to this ODE that involves Whittaker functions :

$$\begin{aligned}
U(a,b;\tau) = & \ \Phi \exp\left[-\frac{1}{2}(\lambda - \alpha)\tau\right] M\left(-\frac{\sqrt{2}}{4}\frac{a\sigma}{\alpha\sqrt{b}}, \frac{1}{2}\frac{\lambda}{\alpha}, \frac{\sigma\sqrt{2b}}{\alpha}e^{-\alpha\tau}\right) \\
& + \Psi \exp\left[-\frac{1}{2}(\lambda - \alpha)\tau\right] W\left(-\frac{\sqrt{2}}{4}\frac{a\sigma}{\alpha\sqrt{b}}, \frac{1}{2}\frac{\lambda}{\alpha}, \frac{\sigma\sqrt{2b}}{\alpha}e^{-\alpha\tau}\right)
\end{aligned} \tag{27}$$

where $M(.)$ and $W(.)$ are Whittaker functions and constants[8] Φ and Ψ can be determined using the initial conditions in ODE (26).

To compute the cross-moments, we must compute the derivatives of functions D and C w.r.t variables a and b that appear in the first and the third arguments in Whittaker functions as well as in Φ and Ψ, which is very heavy. Instead, we will develop a Frobenius[9] solution to the ODE (26) (see Appendix 3.B for details). Making the change of variable $z = \exp(-\alpha\tau)$ and defining the function $S(z)$ by :

$$\begin{cases} S(z) \equiv z^{-\beta} U(\tau) \\[2ex] \beta = \dfrac{\lambda - \alpha}{2\alpha} \end{cases} \tag{28}$$

leads to the following series solution :

$$S(z) = \Phi \sum_{n=0}^{+\infty} k_n(\beta + 1)z^{n+\beta+1} + \Psi \sum_{n=0}^{+\infty} k_n(-\beta)z^{n-\beta} \tag{29}$$

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[8] Φ and Ψ will depend on the variables a and b.

[9] Under some regularity conditions, an ODE may have a series solution taking the form $z^\varepsilon \sum_{n=0}^{+\infty} k_n z^n$.

where constants Φ and Ψ are determined using the fact that $S(1) = 1$ and $S'(1) = -\beta$, and functions $(k_n(.))_{0 \leq n}$ are computed recursively by the following formulas :

$$
\begin{cases}
k_1(\varepsilon) = \dfrac{a\sigma^2}{4\alpha^2 \varepsilon} k_0 \\[2ex]
k_n(\varepsilon) = \dfrac{\sigma^2}{2\alpha^2} \dfrac{(ak_{n-1}(\varepsilon) + bk_{n-2}(\varepsilon))}{(n^2 + n(2\varepsilon - 1))} \quad ; \quad n \geq 2
\end{cases}
\tag{30}
$$

where k_0 is an arbitrary constant. Once functions D and C are computed according to Equation (3.25), the computation of the moments can be achieved by differentiating the function g in Equation (3.20). We only need to differentiate the series solution in Equation (3.29) by truncating it at a finite order instead of dealing with Whittaker functions in Equation (3.27). The computation of the cross-(centered)-moments in Equation (3.18) is thus straightforward.

The no mean reversion case

In the case of no mean reversion (i.e. $\alpha = 0$), the function η in PDE (21) becomes constant and the moment generating function[10] is defined by :

$$g(\eta;t,T,V) \equiv E\left(\exp\left[-\eta\int_t^T V_s\,ds\right]\right) \qquad (31)$$

We do not need a Frobenius series solution since the moment generating function g is given by a simple closed-form expression [see Cox, Ingersoll and Ross (1985)] :

$$g(\tau,V) = \exp\big(VD(\tau) + C(\tau)\big) \qquad (32)$$

where

$$\begin{cases} D(\tau) = \dfrac{-2\eta}{\lambda+\omega}\dfrac{1-e^{-\omega\tau}}{1+\delta\,e^{-\omega\tau}} \\[2mm] C(\tau) = \dfrac{-2\kappa\theta}{\sigma^2}\log\left(\dfrac{1+\delta\,e^{-\omega\tau}}{1+\delta}\right) - \dfrac{\kappa\theta}{\sigma^2}(\omega-\lambda)\tau \\[2mm] \omega = \sqrt{\lambda^2 + 2\eta\sigma^2} \quad ; \quad \delta = \dfrac{\omega-\lambda}{\omega+\lambda} \end{cases} \qquad (33)$$

Notice that the solution U to the ODE (26) is also very easy to compute by :

$$U(\eta;\tau) = \frac{1}{2}\frac{\lambda+\omega}{\omega}\exp\left(\frac{1}{2}(\lambda-\omega)\tau\right) + \frac{1}{2}\frac{\omega-\lambda}{\omega}\exp\left(-\frac{1}{2}(\lambda+\omega)\tau\right) \qquad (34)$$

These formulas will be used later for pricing some exotic derivatives on equities. At this stage, we are able to use Equation (3.18) to compute the approximate price for any contingent claim using a Taylor expansion as long as we can compute the derivatives of the pseudo-price in Equation (3.16) w.r.t. average variances either analytically or numerically.

The next section will present some standard and some exotic options on both mean reverting and non-mean reverting underlying assets. The computation of standard option prices in a stochastic volatility model will allow us to check for the accuracy of the series expansion in simple cases where (semi)-closed-form solutions exist, among which Heston (1993) and Tahani (2004) models[11].

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[10] It can be seen as a zero-coupon price by considering $\left(\eta V_s\right)_{t\le s\le T}$ as the instantaneous interest rate in Cox, Ingersoll and Ross (1985) model.

[11] The constant volatility counterparts of Heston (1993) and Tahani (2004) models are respectively Black and Scholes (1973) and Longstaff and Schwartz (1995) models, which will be used to compute the pseudo-prices in Equation (3.16).

III Valuation formulas for exotic options

In this section, we will remind some well-known closed-form pricing formulas for path-dependent stock options under a constant volatility model, which will be used in the series expansion for pricing the same path-dependent options under the square-root stochastic volatility model. In the case of a mean reverting asset and a constant volatility, Leblanc and Scaillet (1998) propose closed-form solutions (up to an inversion of Fourier transform) for some path-dependent options on affine yields among which the arithmetic average option. This approach will be used to derive a pricing formula for credit spread average options. We also will use the distribution of the first passage time for an Ornstein-Uhlenbeck process to a boundary derived in Leblanc et al. (2000) to price digital barrier credit spread options. Once theses pseudo-prices of exotic options are computed, we will use them in Equation (3.18) in order to price the same exotics under a stochastic volatility model.

Barrier and digital asset-or-nothing stock option

These formulas are derived in details in Reiner and Rubinstein (1991) and Haug (1997). The diffusion of the state variable X under the filtration $(\mathcal{G}_t)$ and the average variance $\overline{V}$ are given in Equation (3.11). The stock price is given by e^X and the pseudo-price of a down-out call with strike price K and barrier L when $K \leq L$ is given by :

$$C_{DO}\left(X_t, \overline{V}, K, L, r, t, T\right) = e^{X_t} N(d) - K e^{-r(T-t)} N\left(d - \sqrt{\overline{V}(T-t)}\right)$$

$$- e^{X_t}\left(\frac{L}{e^{X_t}}\right)^{2\psi} N\left(2\psi\sqrt{\overline{V}(T-t)} - d\right) \qquad (35)$$

$$+ K e^{-r(T-t)}\left(\frac{L}{e^{X_t}}\right)^{2\psi-2} N\left((2\psi-1)\sqrt{\overline{V}(T-t)} - d\right)$$

where

$$\psi = \frac{r}{\overline{V}} + \frac{1}{2} \qquad ; \qquad d = \frac{\ln\left(e^{X_t}/L\right)}{\sqrt{\overline{V}(T-t)}} + \psi\sqrt{\overline{V}(T-t)} \qquad (36)$$

and $N(\cdot)$ is the standard normal cumulative function.

The pseudo-price of a digital down-out asset-or-nothing option can be obtained by taking a strike price equal to $K = 0$ in Equation (3.36) :

$$Dig_{DO}\left(X_t,\overline{V},L,r,t,T\right)=e^{X_t}\,\mathrm{N}(d)-e^{X_t}\left(\frac{L}{e^{X_t}}\right)^{2\psi}\mathrm{N}\left(2\psi\sqrt{\overline{V}(T-t)}-d\right) \qquad (37)$$

while the pseudo-price of an up-out put with $K \le L$ is given by :

$$P_{UO}\left(X_t,\overline{V},K,L,r,t,T\right)=-e^{X_t}\,\mathrm{N}(-d)+Ke^{-r(T-t)}\mathrm{N}\left(-d+\sqrt{\overline{V}(T-t)}\right)$$

$$+e^{X_t}\left(\frac{L}{e^{X_t}}\right)^{2\psi}\mathrm{N}\left(d-2\psi\sqrt{\overline{V}(T-t)}\right) \qquad (38)$$

$$-Ke^{-r(T-t)}\left(\frac{L}{e^{X_t}}\right)^{2\psi-2}\mathrm{N}\left(d-(2\psi-1)\sqrt{\overline{V}(T-t)}\right)$$

Digital cash-or-nothing credit spread option

In the case of mean reversion, i.e. $\alpha \ne 0$, the diffusion of the state variable X under the filtration $(\mathcal{G}_t)$ is an Ornstein-Uhlenbeck process given in Equation (3.14) and the average variances $\left(\overline{V_1},\overline{V_2}\right)$ are given in Equation (3.15) :

$$dX_s=\left(\mu-\gamma\overline{V_1}-\alpha X_s\right)ds+\sqrt{\overline{V_2}}\,dW_s \qquad (39)$$

Defining the first hitting time $T_L=\inf\left\{s:X_s\ge L\right\}$, the density of T_L is derived in Leblanc et al. (2000) :

$$\chi\left(\overline{V_1},\overline{V_2};L,s\right)\equiv Q\left(T_L\in ds\right) \qquad (40)$$

$$=\frac{(L-X)}{\sqrt{2\pi}}\exp\left(\frac{\alpha}{2\overline{V_2}}\left[\begin{array}{c}\left(X-\frac{\mu-\gamma\overline{V_1}}{\alpha}\right)^2-\left(L-\frac{\mu-\gamma\overline{V_1}}{\alpha}\right)^2\\+s-(L-X)^2\coth\left(\frac{\alpha}{\overline{V_2}}s\right)\end{array}\right]\right)\times\left(\frac{\alpha}{\overline{V_2}}\frac{1}{\sinh\left(\frac{\alpha}{\overline{V_2}}s\right)}\right)^{\frac{3}{2}}$$

Göing-Jaeschke and Yor (2003) point out that the distribution given in Leblanc et al. (2000) is only true if the boundary level coincides with the long-run mean[12], which in our case leads to $L=\dfrac{\mu-\gamma\overline{V_1}}{\alpha}$ in Equation (3.40). A digital up-in cash-or-nothing credit spread option pays off a certain amount K at maturity if the credit spread e^X crosses the barrier e^L from below during the option life. Its pseudo-price is then given by :

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[12] Linetsky (2003) derives an approximation to this density for any fixed boundary level in terms of Sturm-Liouville eigenfunction expansion, while Alili et al. (????) provide an integral representation.

$$Dig_{UI}\left(X_t, \overline{V_1}, \overline{V_2}, L, r, t, T\right) = e^{-r(T-t)} K \int_t^T \chi\left(\overline{V_1}, \overline{V_2}; L, s\right) ds \qquad (41)$$

In order to apply the series expansion in Equation (3.18), the derivatives of the pseudo-price w.r.t. $\left(\overline{V_1}, \overline{V_2}\right)$ are computed by differentiating under the integral sign.

Option on average credit spread

Denoting the average credit spread by $Y_T = \int_t^T X_s ds$ where the diffusion of X is given in Equation (3.39), the pseudo-price of a call on average credit spread with strike K can be obtained by :

$$
\begin{aligned}
C_{Ave}(\overline{V_1}, \overline{V_2}) &\equiv e^{-r(T-t)} E\left(\left(\exp\left(\frac{Y_T}{T-t}\right) - K\right)^+\right) \\
&= e^{-r(T-t)} \int_{(T-t)\ln(K)}^{+\infty} \left(\exp(\frac{y}{T-t}) - K\right) Q(Y_T \in dy)
\end{aligned}
\qquad (42)
$$

where the density of Y_T can be computed by inverting its characteristic function[13] and then the integration in Equation (3.42) will be done numerically using Gaussian quadrature [see Tahani (2004)].

IV Numerical results

In order to assess the efficiency and the accuracy of the proposed methodology, we price some plain vanilla and exotic options in Heston (1993) and Tahani (2004) square-root stochastic volatility frameworks.

For standard options, Heston (1993) and Tahani (2004) option prices will be considered as the true prices towards which the series expansion must converge. Tables 1 to 4 show the results for standard call options. It is found that the numerical prices converge rapidly to the true prices; at most, we need the 4th order to achieve a good accuracy.

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[13] The characteristic function of $Y_T = e^{\int_t^T X_s ds}$ is given by $E_t\left(e^{\phi Y_T}\right) = \int_{-\infty}^{+\infty} \exp(\phi y) Q(Y_T \in dy)$ and can be computed as a zero-coupon bond price in Vasicek (1977) model where the instantaneous rate is (ϕX_s).

For exotic options, since there are no closed-form formulas under stochastic volatility, we will compute a Monte Carlo price with 10^5 paths using the antithetic variate method and consider it as the true price. Tables 5 to 10 show the results for barrier and digital barrier options[14]. The numerical prices are shown to converge rapidly at low expansion orders, which proves the accuracy of the series method even for exotic options.

The series expansion is also found to be very efficient. In fact, once the (cross)-centered moments are computed for a given set of parameters, they can be used to price as many options as we want, simply by adding terms in the series expansion, which makes the computation time very small.

V Conclusion

We propose an analytical approximation using Taylor series expansion to price exotic options on stocks and on general mean reverting assets when the volatility is stochastic. The main purpose of the expansion method is the computation of the moments of the average variances in a square-root volatility model, which is done using either a closed-form formula if there is no mean reversion; or using a Frobenius series solution in the case of a mean reverting process. The series expansion method is found to be very accurate and very efficient when pricing stock and credit spread standard and exotic options.

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[14] To compare the barrier and digital barrier options approximated prices with Monte Carlo ones, the barrier levels are adjusted to account for the discrete sampling as suggested in Broadie et al. (1997).

Appendix 3.A :

Assuming that the diffusion processes for the state variable and its volatility are given by:

$$dX_u = \left(\mu - \alpha X_u - \gamma V_u\right)dt + \sqrt{V_u}\left(\rho dB_u + \sqrt{1-\rho^2}\,dW_u\right) \tag{A.1}$$

$$dV_u = \left(\kappa\theta - \lambda V_u\right)dt + \sigma\sqrt{V_u}\,dB_u \tag{A.2}$$

the solution for X can be solved as :

$$X_T = e^{-\alpha(T-t)}X_t + \mu\int_t^T e^{-\alpha(T-s)}ds - \gamma\int_t^T e^{-\alpha(T-s)}V_s\,ds$$

$$+ \rho\int_t^T e^{-\alpha(T-s)}\sqrt{V_s}\,dB_s + \sqrt{1-\rho^2}\int_t^T e^{-\alpha(T-s)}\sqrt{V_s}\,dW_s \tag{A.3}$$

By defining the process Y :

$$Y_{t,T} = \rho\int_t^T e^{-\alpha(T-s)}\sqrt{V_s}\,dB_s - \frac{1}{2}\rho^2\int_t^T e^{-2\alpha(T-s)}V_s\,ds \tag{A.4}$$

this allows for the solution X to be rewritten as :

$$X_T = e^{-\alpha(T-t)}X_t + Y_{t,T} + \mu\int_t^T e^{-\alpha(T-s)}ds - \gamma\int_t^T e^{-\alpha(T-s)}V_s\,ds$$

$$+ \frac{1}{2}\rho^2\int_t^T e^{-2\alpha(T-s)}V_s\,ds + \sqrt{1-\rho^2}\int_t^T e^{-\alpha(T-s)}\sqrt{V_s}\,dW_s \tag{A.5}$$

and shows that the effective variable $\tilde{X}$ given by :

$$\tilde{X}_t = X_t + e^{\alpha(T-t)}Y_{t,T} \tag{A.6}$$

solves the following SDE :

$$d\tilde{X}_u = \left(\mu - \gamma\overline{V_1} + \frac{1}{2}\rho^2\overline{V_3} - \alpha\tilde{X}_u\right)du + \sqrt{\left(1-\rho^2\right)\overline{V_2}}\,dW_u \tag{A.7}$$

where the average variances $\overline{V_1}, \overline{V_2}$ and $\overline{V_3}$ are defined :

$$\begin{cases} \overline{V_1}\int_t^T e^{-\alpha(T-s)}ds = \int_t^T e^{-\alpha(T-s)}V_s\,ds \\[2ex] \overline{V_2}\int_t^T e^{-2\alpha(T-s)}ds = \int_t^T e^{-2\alpha(T-s)}V_s\,ds \\[2ex] \overline{V_3}\int_t^T e^{-\alpha(T-s)}ds = \int_t^T e^{-2\alpha(T-s)}V_s\,ds \end{cases} \tag{A.8}$$

Appendix 3.B: Frobenius series solution

The function S given in Equation (3.28) by :

$$\begin{cases} S(z) \equiv z^{-\beta} U(\tau) \\ \beta = \dfrac{\lambda - \alpha}{2\alpha} \end{cases} \tag{B.1}$$

where $z = \exp(-\alpha\tau)$ must solve the following second-order ODE :

$$\begin{cases} z^2 S''(z) - \left(\beta(\beta+1) + \dfrac{a\sigma^2}{2\alpha^2} z + \dfrac{b\sigma^2}{2\alpha^2} z^2 \right) S(z) = 0 \\ S(1) = 1 \quad , \quad S'(1) = -\beta \end{cases} \tag{B.2}$$

Developing a Frobenius series solution for ODE (B.2) consists in finding a special solution

taking the form $\displaystyle\sum_{n=0}^{+\infty} k_n z^{n+\varepsilon}$. Substituting this series in ODE (B.2) gives :

$$z^2 \sum_{n=0}^{+\infty} k_n (n+\varepsilon)(n+\varepsilon-1) z^{n+\varepsilon-2} - \left(\beta(\beta+1) + \dfrac{a\sigma^2}{2\alpha^2} z + \dfrac{b\sigma^2}{2\alpha^2} z^2 \right) \sum_{n=0}^{+\infty} k_n z^{n+\varepsilon} = 0 \tag{B.3}$$

or equivalently :

$$\sum_{n=0}^{+\infty} k_n \left((n+\varepsilon)(n+\varepsilon-1) - \beta(\beta+1) \right) z^{n+\varepsilon} - \sum_{n=1}^{+\infty} \dfrac{a\sigma^2}{2\alpha^2} k_{n-1} z^{n+\varepsilon} - \sum_{n=2}^{+\infty} \dfrac{b\sigma^2}{2\alpha^2} k_{n-2} z^{n+\varepsilon} = 0 \tag{B.4}$$

Comparing the series coefficients yields to :

$$\begin{cases} k_0 \left(\varepsilon(\varepsilon-1) - \beta(\beta+1) \right) = 0 \\ k_1 \left((1+\varepsilon)\varepsilon - \beta(\beta+1) \right) - \dfrac{a\sigma^2}{2\alpha^2} k_0 = 0 \\ k_n \left((n+\varepsilon)(n+\varepsilon-1) - \beta(\beta+1) \right) - \dfrac{a\sigma^2}{2\alpha^2} k_{n-1} - \dfrac{b\sigma^2}{2\alpha^2} k_{n-2} = 0 \quad , n > 1 \end{cases} \tag{B.5}$$

Thus, ODE (B.2) has a Frobenius series solution only if $\varepsilon = \beta+1$ or $\varepsilon = -\beta$, in which

cases, the series coefficients are given by :

$$\begin{cases} k_0 \text{ arbitrary constant} \\ k_1(\varepsilon) = \dfrac{a\sigma^2}{4\alpha^2 \varepsilon} k_0 \\ k_n(\varepsilon) = \dfrac{\sigma^2}{2\alpha^2} \dfrac{ak_{n-1}(\varepsilon) + bk_{n-2}(\varepsilon)}{(n^2 + (2\varepsilon-1)n)} \quad , n \geq 2 \end{cases} \tag{B.6}$$

A general solution to ODE (B.2) is then given by :

$$S(z) = \Phi \sum_{n=0}^{+\infty} k_n (\beta + 1) z^{n+\beta+1} + \Psi \sum_{n=0}^{+\infty} k_n (-\beta) z^{n-\beta} \qquad \text{(B.7)}$$

where the constants Φ and Ψ are determined using the conditions $S(1) = 1$ and $S'(1) = -\beta$. The arbitrary constant k_0 can be taken equal to 1.

References

Alili L., P. Patie, and J.L. Pedersen, 2004, "Representation of the first Hitting Time Density of an Ornstein-Uhlenbeck Process," Working paper, RiskLab.

Apel T., G. Winkler, and U. Wystup, 2001, "Valuation of Options in Heston's Stochastic Volatility Model Using Finite Element Methods," Foreign Exchange Risk, Risk Publications.

Bakshi G., C. Cao, and Z. Chen, 1997, "Empirical Performance of Alternative Option Pricing Models," The Journal of Finance, Vol 52, 2003-2049.

Ball C., and A. Roma, 1994, "Stochastic Volatility Option Pricing," Journal of Financial and Quantitative Analysis, Vol 29, 589-607.

Black F., and M. Scholes, 1973, "The Valuation of Options and Corporate Liabilities," Journal of Political Economy, Vol 81, 637-654.

Broadie M., P. Glasserman, and S. Kou, 1997, "A continuity correction for discrete barrier options, " Mathematical Finance, Vol 7, 325-349.

Clewlow L. J., and C. R. Strickland, 1997, "Monte Carlo Valuation of Interest Rate Derivatives under Stochastic Volatility," The Journal of Fixed Income, Vol 7, 35-45.

Cox J., J. Ingersoll, and S. Ross, 1985, "A Theory of the Term Structure of Interest Rates," Econometrica, Vol 53, 385-407.

Davydov D., and V. Linetsky, 2001, "Pricing and Hedging Path-Dependent Options Under the CEV Process," Management Science, Vol 47, 949-965.

Fong H.G., and O. A. Vasicek, 1992, "Interest Rate Volatility as a Stochastic Factor," Working paper, Gifford Associates.

Göing-Jaschke A., and M. Yorc, 2003, "A Clarification Note about Hitting Times Densities for Ornstein-Uhlenbeck Processes," Finance and Stochastics, Vol 7, 413-415.

Haug E.G., 1997, The Complete Guide to Option Pricing Formulas, McGraw-Hill.

Henderson V., and D. Hobson, 2000, "Passport Options with Stochastic Volatility," Working paper, University of Bath.

Heston S.L., 1993, "A Closed-form Solution for Options with Stochastic Volatility, with Applications to Bond and Currency Options," The Review of Financial Studies, Vol 6, 327-343.

Hull J., and A. White, 1987, "The Pricing of Options on Assets with Stochastic Volatilities," Journal of Finance, Vol 42, 281-300.

Karatzas I., and S.A. Shreve, 1991. Brownian Motion and Stochastic Calculus, Springer Verlag, New York.

Jacobs K., and X. Li, 2004, "Modeling the Dynamics of Credit Spreads with Stochastic Volatility," Finance and Stochastics, Vol 2, 349-367.

Leblanc B., and O. Scaillet, 1998, "Path Dependent Options on Yields in the Affine Term Structure Model," Finance and Stochastics, Vol 2, 349-367.

Leblanc B., and O. Scaillet, 2000, "A Correction Note on the First Passage Time of an Ornstein-Uhlenbeck Process to a Boundary," Finance and Stochastics, Vol 4, 109-111.

Lewis A., 2000, Option Valuation under Stochastic Volatility, Finance press, California.

Linetsky V., 2003, "Computing Hitting Time Densities for OU and CIR Processes: Applications to Mean-reverting Models, " Working paper, Department of Industrial Engineering and Management Sciences, Northwestern University.

Longstaff F.A., and E.S. Schwartz, 1995, "Valuing Credit Derivatives," The Journal of Fixed Income, Vol 5, 6-12.

Prigeant J.-L., O. Renault, and O. Scaillet, 2001, "An Empirical Investigation into Credit Spread Indices," Journal of Risk, Vol 3, 27-55.

Reiner E., and M. Rubinstein, 1991, "Breaking down the barriers," Risk, Vol 4, 28-35.

Romano M., and N. Touzi, 1997, "Contingent Claims and Market Completeness in a Stochastic Volatility Model," Mathematical Finance, Vol 7, 339-412.

Sabanis S., 2002, "Stochastic volatility," International Journal of Theoretical and Applied Finance, Vol 5, 515-530.

Sabanis S., 2003, "Stochastic volatility and the Mean Reverting Process," The Journal of Futures Markets, Vol 23, 33-47.

Schöbel R., and J. Zhu, 1999, "Stochastic Volatility With an Ornstein-Uhlenbeck Process: An Extension," European Finance Review, Vol 3, 23-46.

Schwartz E.S., 1997, "The Stochastic Behavior of Commodity Prices: Implications for Valuation and Hedging," The Journal of Finance, Vol 3, 6-12.

Selby M. J. P., and C. R. Strickland, 1995, "Computing the Fong and Vasicek Pure Discount Bond Price Formula," The Journal of Fixed Income, Vol 5, 78-84.

Stein E., and J. Stein, 1991, "Stock Price Distributions with Stochastic Volatility: An Analytic Approach," The Review of Financial Studies, Vol 4, 727-752.

Tahani N., 2000, "Credit Spread Option Valuation under GARCH," Working paper 00-07, Canada Research Chair in Risk Management, HEC Montréal. http://www.hec.ca/gestiondesrisques/00-07.pdf.

Tahani N., 2004, "Valuing Credit Derivatives Using Gaussian Quadrature: A Stochastic Volatility Framework," The Journal of Futures Markets, Vol 24, 3-35.

Vasicek O.A., 1977, "An Equilibrium Characterization of the Term Structure," Journal of Financial Economics, Vol 5, 177-188.

Zhu J., 2000, "Modular Pricing of Options," Working paper, Eberhard-Karls-Universität Tübingen.

Tables

Table 3.1 : Plain vanilla call price under Heston model

Order	Price	Absolute error	Relative error
0	8,1802	1,27e-02	1,55e-03
2	8,1674	1,53e-04	1,88e-05
3	8,1677	2,03e-04	2,49e-05
4	8,1675	1,70e-05	2,08e-06
5	8,1675	9,43e-06	1,16e-06
6	8,1675	1,84e-06	2,25e-07
7	8,1675	7,31e-07	8,94e-08
8	8,1675	1,83e-07	2,25e-08
9	8,1675	1,82e-07	2,23e-08
10	8,1675	4,26e-08	5,21e-09
True price	8,1675		

Table 3.1 presents the results of the valuation of a plain vanilla call within Heston (1993) model for different expansion orders. The true price is given by Heston closed-form formula. The option's parameters are $X_0 = \ln(100)$; $V_0 = 0.04$; $K = 100$; $r = 0.05$; $T = 1$. The model's parameters are $\mu = r$; $\alpha = 0$; $\gamma = 0.5$; $\sigma = 0.1$; $\lambda = 4$; $\kappa = 1$ and $\theta = 0.05$.

Table 3.2 : Plain vanilla call price under Heston model

Order	Price	Absolute error	Relative error
0	15,2350	1,95e-03	1,28e-04
2	15,2369	2,60e-05	1,71e-06
3	15,2367	2,28e-04	1,49e-05
4	15,2369	2,29e-05	1,50e-06
5	15,2369	1,53e-05	1,00e-06
6	15,2369	3,21e-06	2,11e-07
7	15,2369	1,19e-06	7,82e-08
8	15,2369	2,98e-07	1,95e-08
9	15,2369	4,88e-07	3,20e-08
10	15,2369	2,77e-08	1,82e-09
True price	15,2369		

Table 3.2 presents the results of the valuation of a plain vanilla call within Heston (1993) model for different expansion orders. The true price is given by Heston closed-form formula. The option's parameters are $X_0 = \ln(100)$; $V_0 = 0.04$; $K = 90$; $r = 0.05$; $T = 1$. The model's parameters are $\mu = r$; $\alpha = 0$; $\gamma = 0.5$; $\sigma = 0.1$; $\lambda = 4$; $\kappa = 1$ and $\theta = 0.05$.

Table 3.3 : Plain vanilla call price under Tahani model

Order	Price	Absolute error	Relative error
0	0,0019759	2,08e-06	1,05e-03
2	0,0019738	4,71e-08	2,39e-05
3	0,0019738	2,17e-09	1,10e-06
4	0,0019738	2,04e-08	1,03e-05
5	0,0019738	1,89e-08	9,58e-06
6	0,0019738	1,88e-08	9,53e-06
7	0,0019738	1,91e-08	9,67e-06
8	0,0019738	1,88e-08	9,54e-06
9	0,0019738	1,90e-08	9,62e-06
10	0,0019738	1,89e-08	9,56e-06
True price	0,0019738		

Table 3.3 presents the results of the valuation of a plain vanilla call within Tahani (2004) square-root model for different expansion orders. The true price is given by Tahani semi-closed-form formula. The option's parameters are $X_0 = \ln(0.02)$; $V_0 = 0.04$; $K = 0.02$; $r = 0.05$; $T = 1$. The model's parameters are $\mu = 0.03$; $\alpha = 0.01$; $\gamma = 0.2$; $\sigma = 0.1$; $\lambda = 4$; $\kappa = 1$ and $\theta = 0.05$.

Table 3.4 : Plain vanilla call price under Tahani model

Order	Price	Absolute error	Relative error
0	0,0066518	4,02e-06	6,04e-04
2	0,0066561	2,89e-07	4,34e-05
3	0,0066559	3,74e-08	5,62e-06
4	0,0066557	1,75e-07	2,62e-05
5	0,0066559	7,57e-08	1,14e-05
6	0,0066557	1,57e-07	2,36e-05
7	0,0066558	2,72e-08	4,08e-06
8	0,0066558	2,45e-08	3,69e-06
9	0,0066556	2,16e-07	3,25e-05
10	0,0066561	2,48e-07	3,72e-05
True price	0,0066558		

Table 3.4 presents the results of the valuation of a plain vanilla call within Tahani (2004) square-root model for different expansion orders. The true price is given by Tahani semi-closed-form formula. The option's parameters are $X_0 = \ln(0.025)$; $V_0 = 0.04$; $K = 0.02$; $r = 0.05$; $T = 1$. The model's parameters are $\mu = 0.03$; $\alpha = 0.01$; $\gamma = 0$; $\sigma = 0.2$; $\lambda = 4$; $\kappa = 1$ and $\theta = 0.05$.

Table 3.5 : Down-out barrier call price under Heston model

Order	Price	Absolute error	Relative error
0	8,4046	1,79e-02	2,12e-03
2	8,4226	1,40e-04	1,66e-05
3	8,4221	4,09e-04	4,85e-05
4	8,4226	1,54e-04	1,83e-05
5	8,4226	9,69e-05	1,15e-05
6	8,4226	1,29e-04	1,54e-05
7	8,4226	1,23e-04	1,46e-05
8	8,4226	1,26e-04	1,49e-05
9	8,4226	1,25e-04	1,49e-05
10	8,4226	1,25e-04	1,49e-05
Monte Carlo	8,4225		

Table 3.5 presents the results of the valuation of a down-out barrier call within Heston (1993) model for different expansion orders. The Monte Carlo price is given by 10^5 paths using the antithetic variate method. The option's parameters are $X_0 = \ln(100)$; $V_0 = 0.04$; $K = 90$; $L = 95$; $r = 0.05$; $T = 0.36$. The model's parameters are $\mu = r$; $\alpha = 0$; $\gamma = 0.5$; $\sigma = 0.1$; $\lambda = 4$; $\kappa = 1$ and $\theta = 0.05$.

Table 3.6 : Down-out barrier call price under Heston model

Order	Price	Absolute error	Relative error
0	9,3847	3,17e-02	3,37e-03
2	9,4168	4,23e-04	4,49e-05
3	9,4153	1,10e-03	1,17e-04
4	9,4166	2,35e-04	2,50e-05
5	9,4164	2,67e-05	2,83e-06
6	9,4165	1,37e-04	1,46e-05
7	9,4165	1,07e-04	1,13e-05
8	9,4165	1,20e-04	1,27e-05
9	9,4165	1,15e-04	1,23e-05
10	9,4165	1,17e-04	1,24e-05
Monte Carlo	9,4164		

Table 3.6 presents the results of the valuation of a down-out barrier call within Heston (1993) model for different expansion orders. The Monte Carlo price is given by 10^5 paths using the antithetic variate method. The option's parameters are $X_0 = \ln(100)$; $V_0 = 0.04$; $K = 90$; $L = 95$; $r = 0.05$; $T = 1$. The model's parameters are $\mu = r$; $\alpha = 0$; $\gamma = 0.5$; $\sigma = 0.1$; $\lambda = 4$; $\kappa = 1$ and $\theta = 0.05$.

Table 3.7 : Up-out barrier put price under Heston model

Order	Price	Absolute error	Relative error
0	1,8159	1,30e-02	7,19e-03
2	1,7992	3,80e-03	2,11e-03
3	1,7994	3,55e-03	1,97e-03
4	1,7993	3,70e-03	2,05e-03
5	1,7993	3,70e-03	2,05e-03
6	1,7993	3,69e-03	2,05e-03
7	1,7993	3,69e-03	2,05e-03
8	1,7993	3,69e-03	2,05e-03
9	1,7993	3,69e-03	2,05e-03
10	1,7993	3,69e-03	2,05e-03
Monte Carlo	1,8030		

Table 3.7 presents the results of the valuation of an up-out barrier put within Heston (1993) model for different expansion orders. The Monte Carlo price is given by 10^5 paths using the antithetic variate method. The option's parameters are $X_0 = \ln(100)$; $V_0 = 0.04$; $K = 100$; $L = 110$; $r = 0.05$; $T = 0.24$. The model's parameters are $\mu = r$; $\alpha = 0$; $\gamma = 0.5$; $\sigma = 0.1$; $\lambda = 4$; $\kappa = 1$ and $\theta = 0.05$.

Table 3.8 : Up-out barrier put price under Heston model

Order	Price	Absolute error	Relative error
0	6,6753	6,20e-02	9,38e-03
2	6,6119	1,41e-03	2,14e-04
3	6,6143	9,68e-04	1,46e-04
4	6,6116	1,70e-03	2,57e-04
5	6,6120	1,34e-03	2,02e-04
6	6,6117	1,58e-03	2,39e-04
7	6,6118	1,51e-03	2,29e-04
8	6,6118	1,55e-03	2,35e-04
9	6,6118	1,54e-03	2,32e-04
10	6,6118	1,55e-03	2,34e-04
Monte Carlo	6,6133		

Table 3.8 presents the results of the valuation of an up-out barrier put within Heston (1993) model for different expansion orders. The Monte Carlo price is given by 10^5 paths using the antithetic variate method. The option's parameters are $X_0 = \ln(100)$; $V_0 = 0.09$; $K = 100$; $L = 120$; $r = 0.05$; $T = 1$. The model's parameters are $\mu = r$; $\alpha = 0$; $\gamma = 0.5$; $\sigma = 0.1$; $\lambda = 1$; $\kappa = 2$ and $\theta = 0.05$.

Table 3.9 : Digital down-out asset-or-nothing option

price under Heston model

Order	Price	Absolute error	Relative error
0	78,7675	1,58e-01	2,00e-03
2	78,9361	1,09e-02	1,38e-04
3	78,9344	9,17e-03	1,16e-04
4	78,9329	7,68e-03	9,73e-05
5	78,9334	8,15e-03	1,03e-04
6	78,9329	7,70e-03	9,76e-05
7	78,9330	7,83e-03	9,92e-05
8	78,9330	7,76e-03	9,83e-05
9	78,9330	7,78e-03	9,86e-05
10	78,9330	7,77e-03	9,84e-05
Monte Carlo	78,9252		

Table 3.9 presents the results of the valuation of a digital down-out asset-or-nothing option within Heston (1993) model for different expansion orders. The Monte Carlo price is given by 10^5 paths using the antithetic variate method. The option's parameters are $X_0 = \ln(100)$; $V_0 = 0.04$; $L = 90$; $r = 0.05$; $T = 0.36$. The model's parameters are $\mu = r$; $\alpha = 0$; $\gamma = 0.5$; $\sigma = 0.1$; $\lambda = 4$; $\kappa = 1$ and $\theta = 0.05$.

Table 3.10 : Digital up-in cash-or-nothing option

price under Tahani model

Order	Price	Absolute error	Relative error
0	0,9393	3,01e-04	3,21e-04
2	0,9397	9,27e-05	9,87e-05
3	0,9396	6,97e-05	7,41e-05
4	0,9397	9,36e-05	9,96e-05
5	0,9396	8,93e-05	9,50e-05
6	0,9397	9,19e-05	9,78e-05
7	0,9397	9,11e-05	9,69e-05
8	0,9397	9,15e-05	9,73e-05
9	0,9397	9,13e-05	9,72e-05
10	0,9397	9,14e-05	9,72e-05
Monte Carlo	0,9396		

Table 3.10 presents the results of the valuation of a digital down-out asset-or-nothing option within Tahani (2004) model for different expansion orders. The Monte Carlo price is given by 10^5 paths using the antithetic variate method. The option's parameters are $X_0 = \ln(0.045)$; $V_0 = 0.04$; $K = 1$; $L = 0.05$; $r = 0$; $T = 1$. The model's parameters are $\mu = -0.03$; $\alpha = 0.01$; $\gamma = 0$; $\sigma = 0.1$; $\lambda = 4$; $\kappa = 1$ and $\theta = 0.05$.