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When Can Expected Utility Handle First-Order Risk Aversion?

Georges Dionne*

Canada Research Chair in Risk Management, HEC Montréal, CIRRELT, and CIRPÉE,

3000 Cote Ste Catherine, Montreal, Qc, Canada, H3T 2A7

Jingyuan Li

Department of Finance and Insurance

Lingnan University

8 Castle Peak Road, Tuen Mun, Hong Kong

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Abstract

Expected utility functions are limited to second-order (conditional) risk aversion, while non-expected utility functions can exhibit either first-order or second-order (conditional) risk aversion. We extend the concept of orders of conditional risk aversion to orders of conditional dependent risk aversion. We show that first-order conditional dependent risk aversion is consistent with the framework of the expected utility hypothesis. Our theoretical result proposes new insights into economic and financial applications such as the equity premium puzzle, the cost of business cycles, and stock market participation. Our model is compared to the rank-dependent expected utility model.

Keywords: Expected utility theory; first-order conditional dependent risk aversion; background risk; equity premium puzzle; consumption risk in business cycles; coward gambler; stock market participation; rank-dependent expected utility model.

JEL classification: D81; G10; G11; G12.

*Corresponding author: Email georges.dionne@hec.ca; Tel 514-340-6596; Fax 514-340-5019.

1 Introduction

We present a new approach to developing an expected utility model that exhibits first-order risk aversion and explains some economic and finance puzzles. The main contribution of our model is to show that dependent background risk is sufficient to obtain a first-order risk premium distinct from a second-order premium with the standard expected utility framework. Because many economic agents are often exposed to simultaneous dependent risks (for example, an IBM employee who is considering buying IBM stock or a non-permanent employee who owns units of stock index and whose job is subject to business cycles) the sufficient condition we propose is natural for many economic agents.

The concepts of second-order and first-order risk aversion were coined by Segal and Spivak (1990). For an actuarially fair random variable $\tilde{\varepsilon}$, second-order risk aversion means that the risk premium the agent is willing to pay to avoid $k\tilde{\varepsilon}$ is proportional to k^2 as $k \to 0$. Under first-order risk aversion, the risk premium is proportional to k. Loomes and Segal (1994) extend this notion to preferences about uninsured events, such as independent additive background risks. They introduce the concept of orders of conditional risk aversion. Defining $\tilde{y}$ as an independent additive risk, the conditional risk premium is the amount of money the decision maker is willing to pay to avoid $\tilde{\varepsilon}$ in the presence of $\tilde{y}$. The preference relation satisfies first-order conditional risk aversion if the risk premium the agent is willing to pay to avoid $k\tilde{\varepsilon}$ is proportional to k as $k \to 0$. It satisfies second-order conditional risk aversion if the risk premium is proportional to k^2.

First-order risk aversion implies that small risks matter. It is well known from Arrow (1974) and Borch (1974) that differentiable expected utility (EU) is only second order. Because expected utility theory is limited to second-order (conditional) risk aversion, it ignores many real world results. Several non-EU models that can predict first-order risk aversion behavior (i.e. rank-dependent EU and loss aversion) are being used in the economic and financial literatures to explain puzzles that EU cannot handle (see, Epstein and Zin 1990; Quiggin 1982; Tversky and Kahneman 1992).

For example, risk-averse individuals are reluctant to take a small actuarially favorable gamble or to invest in financial markets with a positive risk premium. These two puzzles —coward gambler (Samuelson, 1963) and the stock market participation puzzle (Barberis et al., 2006)— have been solved with the property of first-order risk aversion, but outside the expected utility framework. Two other significant puzzles are the equity premium puzzle (Mehra and Prescott, 1985), in which asset prices are poorly explained by reasonable values of risk aversion, and the welfare cost of business cycles considered small by an expected utility representative agent who does not consider dependent background risk (Lucas, 1987).

In this paper, we reinvestigate whether first-order conditional risk aversion appears in the framework of the expected utility hypothesis. The general answer to this question is positive with some weak restrictions: expected utility theory exhibits first-order risk aversion when there is a dependent background risk, but not otherwise. We extend the concepts of orders of conditional risk aversion to orders of conditional dependent risk aversion, for which $\tilde{\varepsilon}$ and the background risk $\tilde{y}$ are dependent and $\tilde{y}$ may enter the agent's utility function arbitrarily. We thus propose a new source of first-order risk aversion over general preferences. Because EU theory is simple, parsimonious, and able to explain a wide set of empirical facts, our use of first-order risk aversion is easy to interpret in many real-world applications, and the assumptions are plausible. It is now well accepted that a labor or income risk that changes over economic cycles can be interpreted as a correlated background risk for a representative agent when choosing optimal portfolio or assets in pension funds.

We propose conditions on the stochastic structure between $\tilde{\varepsilon}$ and $\tilde{y}$ that guarantee first-order conditional dependent risk aversion for expected utility agents with a certain type of risk preference; i.e., with correlation aversion. Eeckhoudt, Rey and Schlesinger (2007) provide an economic interpretation of correlation aversion: a higher level of the background variable mitigates the detrimental effect of a reduction in wealth. It turns out that the concept of expectation dependence proposed by Wright (1987), which is a stronger measure of dependence than covariance, is a key element of such a stochastic structure.

The paper proceeds as follows: Section 2 sets up the model. Section 3 explains the concept of expectation dependence. Section 4 discusses the concept of orders of conditional risk aversion and investigates the orders of conditional dependent risk aversion. It also presents two special cases and two examples of the general model. Section 5 applies the results to two economic puzzles. Section 6 offers a direct comparison between expected utility and rank-dependent expected utility for generating first-order risk aversion. Section 7 concludes the paper.

2 The model

We consider an agent whose preference for random wealth, $\tilde{w}$, and a random outcome, $\tilde{y}$, can be represented by a bivariate expected utility function. Let $u(w, y)$ be the utility function. We assume that all partial derivatives required for any definition exist. We make the standard assumption that $u_1 > 0$, where $u_1 = \frac{\partial u}{\partial w}$.

Let us assume that $\tilde{z} = k\tilde{\varepsilon}$ is the risk faced by a risk-averse agent. Parameter k can be interpreted as the size of the risk. One way to measure an agent's degree of risk aversion for $\tilde{z}$ is to ask him how much he is willing to pay to eliminate $\tilde{z}$. This value will be referred to as the risk premium $\pi(k)$ associated with that risk. For an agent with utility function u, $E\tilde{y}$ the expected value of another risk $\tilde{y}$, and non-random initial wealth w, the risk premium $\pi(k)$ must satisfy the following condition in the absence of a background risk:

$$u(w + Ek\tilde{\varepsilon} - \pi(k), E\tilde{y}) = Eu(w + k\tilde{\varepsilon}, E\tilde{y}). \tag{1}$$

Segal and Spivak (1990) give the following definitions of first- and second-order risk aversion:

Definition 2.1 *(Segal and Spivak, 1990) The agent's attitude towards risk at w is first-order if for every $\tilde{\varepsilon}$ with $E\tilde{\varepsilon} = 0$, $\pi'(0) \neq 0$. The agent's attitude towards risk at w is second-order if for every $\tilde{\varepsilon}$ with $E\tilde{\varepsilon} = 0$, $\pi'(0) = 0$ but $\pi''(0) \neq 0$.*

They provide the following results linking properties of the order of expected utility model to its order of risk aversion given the level of wealth w_0:

(a) If the utility function u is not differentiable at w_0 but has well-defined and distinct left and right derivatives at w_0, then the agent exhibits first-order risk aversion at w_0.

(b) If the utility function u is twice differentiable at w_0 with $u_{11} < 0$, then the agent exhibits second-order risk aversion at w_0.

Segal and Spivak (1996) point out that if the von Neumann-Morgenstern utility function is increasing, then it must be differentiable almost everywhere, and one may convincingly argue that non-differentiability is seldom observed in the expected utility model. Alternatively, like increasing functions, concave functions may still have a countable set of points of non-differentiability.

Loomes and Segal (1994) introduced the order of conditional risk aversion by examining the characteristics of $\pi(k)$ in the presence of an independent uninsured risk $\tilde{y}^i$. For an agent with utility function u and initial wealth w, the conditional risk premium $\pi_c(k)$ must satisfy the following condition:

$$Eu(w + Ek\tilde{\varepsilon} - \pi_c(k), \tilde{y}^i) = Eu(w + k\tilde{\varepsilon}, \tilde{y}^i). \tag{2}$$

Definition 2.2 *(Loomes and Segal, 1994) The agent's attitude towards risk at w is first-order conditional risk aversion if for every $\tilde{\varepsilon}$ with $E\tilde{\varepsilon} = 0$, $\pi_c'(0) \neq 0$. The agent's attitude towards risk at w is second-order conditional risk aversion if for every $\tilde{\varepsilon}$ with $E\tilde{\varepsilon} = 0$, $\pi_c'(0) = 0$ but $\pi_c''(0) \neq 0$.*

It is obvious that the definitions of first- and second-order conditional risk aversion are more general than the definitions of first- and second-order risk aversion. We can extend the above definitions to the case $E\tilde{\varepsilon} \neq 0$. Because u is differentiable, fully differentiating (2) with respect to k yields:

$$E\{[E\tilde{\varepsilon} - \pi_c'(k)]u_1(w + Ek\tilde{\varepsilon} - \pi_c(k), \tilde{y}^i)\} = E[\tilde{\varepsilon}u_1(w + k\tilde{\varepsilon}, \tilde{y}^i)]. \tag{3}$$

Because $\tilde{\varepsilon}$ and $\tilde{y}^i$ are independent,

$$\pi_c'(0) = \frac{E\tilde{\varepsilon}Eu_1(w, \tilde{y}^i) - E[\tilde{\varepsilon}u_1(w, \tilde{y}^i)]}{Eu_1(w, \tilde{y}^i)} = 0. \tag{4}$$

Therefore, not only does $\pi_c(k)$ approach zero as k approaches zero, but also $\pi_c'(0) = 0$. This implies that at the margin, accepting a small risk has no effect on the welfare of economic agents. This is an important property of expected utility theory: in the small, the expected utility maximizers are risk-neutral in the presence of an independent background risk.

Using a Taylor expansion of π_c around $k = 0$, we obtain that[1]

$$\pi_c(k) = \pi_c(0) + \pi_c'(0)k + O(k^2) = O(k^2). \tag{5}$$

This result is the Arrow-Pratt approximation, which states that the conditional risk premium is approximately proportional to the square of the size of the risk.

In conclusion, if the random outcome and the background risk are independent, then second-order conditional risk aversion relies on the assumption that the utility function is differentiable. Hence, within a framework of independent background risk, utility functions in the von

[1] In the statistical literature, the sequence b_k is at most of order k^λ, denoted as $b_k = O(k^\lambda)$, if for some finite real number $\Delta > 0$, there exists a finite integer K such that for all $k > K$, $|k^\lambda b_k| < \Delta$ (see White, 2000, p. 16).

Neumann-Morgenstern expected utility class can generically exhibit only second-order conditional risk aversion, and cannot explain the rejection of a small and actuarially favorable gamble or the acceptance of a full insurance contract with actuarially unfair pricing.

3 Expectation Dependence

We denote by $F(\varepsilon, y)$ the joint distribution function of $(\tilde{\varepsilon}, \tilde{y})$. $F_\varepsilon(\varepsilon)$ and $F_y(y)$ are the marginal distributions. Wright (1987) introduced the concept of expectation dependence in the economics literature.

Definition 3.1 *If*

$$ED(y) = [E\tilde{\varepsilon} - E(\tilde{\varepsilon}|\tilde{y} \leq y)] \geq 0 \; for \; all \; y, \tag{6}$$

$$and \; there \; is \; at \; least \; some \; y_0 \; for \; which \; a \; strong \; inequality \; holds,$$

then $\tilde{\varepsilon}$ *is positive expectation dependent on* $\tilde{y}$*. Similarly,* $\tilde{\varepsilon}$ *is negative expectation dependent on* $\tilde{y}$ *if (6) holds with the inequality sign reversed.*

Wright (1987, p. 113) interprets negative expectation dependence as follows: "when we discover $\tilde{y}$ is small, in the precise sense that we are given the truncation $\tilde{y} \leq y$, our expectation of $\tilde{\varepsilon}$ is revised upward." He also provides the following theorem to link expectation dependence and covariance.

Theorem 3.2 *(Wright 1987, Theorem 3.1)* $\tilde{\varepsilon}$ *is positive (negative) expectation dependent on* $\tilde{y}$ *if and only if* $Cov(\tilde{\varepsilon}, m(\tilde{y}) \geq (\leq) 0$ *for every increasing function* m*.*

Hence, "$\tilde{\varepsilon}$ is positive (negative) expectation dependent on $\tilde{y}$" does not mean that "$\tilde{y}$ is the causality of $\tilde{\varepsilon}$." Expectation dependence simply means that the two variables are dependent, and the covariance between the two random variables ($Cov(\tilde{\varepsilon}, \tilde{y})$) is not strong enough to measure this dependence.

One example of distribution with positive expectation dependent random variables is the bivariate normal distribution with positive correlation coefficient. Because positive (negative) expectation dependence is a weaker definition than positive (negative) quadrant dependence (Wright, 1987, p. 114), bivariate random variables that are positive (negative) quadrant dependent (Lehmann, 1966) are also positive (negative) expectation dependent. Portfolio selection problems with positive quadrant dependent (PQD) variables are explored by Pellerey and Semeraro (2005) and Dachraoui and Dionne (2007), among others. Pellerey and Semeraro (2005) assert that a large subset of the elliptical distribution class is PQD. Many other bivariate random variables besides elliptical distributions are PQD. For examples of such distributions, see Joe (1997) and Balakrishnan and Lai (2009). Dachraoui and Dionne (2007) also present examples of PQD bivariate distributions that are not elliptical. Dionne et al. (2012) discuss examples of distributions where the sign of $Cov(\tilde{\varepsilon}, \tilde{y})$ differs from that of $Cov(\tilde{\varepsilon}, m(\tilde{y}))$.

In the following proposition, we introduce a specific distributional assumption that is frequently used in the macroeconomics literature for representing an idiosyncratic risk.

Proposition 3.3 *Suppose $\tilde{y} = h(\tilde{\varepsilon}, \tilde{u})$, where $h(\varepsilon, u)$ is increasing (decreasing) in ε, and $(\tilde{\varepsilon}, \tilde{u})$ are independent random variables, then $\tilde{\varepsilon}$ is positive (negative) expectation dependent on $\tilde{y}$.*

Proof See Appendix.

Suppose $\tilde{y}$ is an idiosyncratic risk and $\tilde{\varepsilon}$ is an aggregate risk. Although $\tilde{y}$ is dependent on $\tilde{\varepsilon}$, Proposition 3.3 shows that the aggregate risk is expectation dependent on the idiosyncratic risk. We use the following two examples to illustrate that Proposition 3.3 corresponds to typical ways of introducing an idiosyncratic risk in macroeconomic and finance literatures.

Example 1 Denote $\tilde{C}$ as aggregate consumption. The labor income (earnings) of worker i is denoted by $\tilde{W}_i$. Labor income is uncertain and defined by $\tilde{W}_i = (aI_{\{\tilde{C} \leq C^*\}} + bI_{\{\tilde{C} > C^*\}})g\tilde{u}_i$, where g, a, b, and C^* are positive constants, $b \geq a$, $I_{\{A\}}$ denotes the indicator function of the event A (equal to 1 if A is realized and to 0 otherwise), $\tilde{u}_i > 0$, and $(\tilde{C}, \tilde{u}_i)$ are independent random variables. For a dynamic version of this example, we refer to Krebs (2007, p. 667). $aI_{\{\tilde{C} \leq C^*\}} + bI_{\{\tilde{C} > C^*\}}$ is the cyclical component of labor income risk, and we can interpret this component as describing job displacement risk. $b - a$ is the income loss of a worker who is displaced when the aggregate state $\tilde{C} \leq C^*$. Because $h(C, u_i) = (aI_{\{C \leq C^*\}} + bI_{\{C > C^*\}})gu_i$ is increasing in C, Proposition 3.3 implies that $\tilde{C}$ is positive expectation dependent on $\tilde{W}_i$.

Example 2 Define $\tilde{C}$ as aggregate per capita consumption. $\tilde{\delta}_i$ is an idiosyncratic permanent income shock for individual i. $\tilde{W}_i = e^{\tilde{\delta}_i}\tilde{C}$ is individual i's labor income. $(\tilde{C}, \tilde{\delta}_i)$ are independent random variables. For a dynamic version, we refer to Constantinides and Duffie (1996) and De Santis (2007). Because $h(C, \delta_i) = e^{\delta_i}C$ is increasing in C, from Proposition 3.3, we know that $\tilde{C}$ is positive (negative) expectation dependent on $\tilde{W}_i$.

Normally one would think that the distribution of idiosyncratic shocks depends on the realization of the aggregate state, but the above two examples describe the reverse relation (i.e. distribution of aggregate risk as a function of idiosyncratic risks), which is not nearly as evident to interpret. However, Theorem 3.2 clarifies that expectation dependence does not mean there is causality but rather correlation between idiosyncratic shocks and the aggregate state. Therefore, the two examples only state that idiosyncratic shocks and the aggregate state are strongly correlated in both directions.

4 Order of conditional dependent risk aversion

4.1 Definition

We now introduce the concept of order of conditional dependent risk aversion. For an agent with utility function u and non-random initial wealth w, the conditional dependent risk premium, $\pi_{cd}(k)$, must satisfy the following condition:

$$Eu(w + Ek\tilde{\varepsilon} - \pi_{cd}(k), \tilde{y}) = Eu(w + k\tilde{\varepsilon}, \tilde{y}), \tag{7}$$

where $\tilde{\varepsilon}$ and $\tilde{y}$ are not necessarily independent risks. We propose the following definitions:

Definition 4.1 *The agent's attitude towards risk at w is first-order conditional dependent risk aversion if for every $\tilde{\varepsilon}$, $\pi_{cd}(k) - \pi_c(k) = O(k)$. The agent's attitude towards risk at w is second-order conditional dependent risk aversion if for every $\tilde{\varepsilon}$, $\pi_{cd}(k) - \pi_c(k) = O(k^2)$.*

$\pi_{cd}(k) - \pi_c(k)$ measures how dependence between risks affects the risk premium. Second-order conditional dependent risk aversion implies that in the presence of a dependent background risk, a small risk has no (significant) effect on risk premium. However, first-order conditional dependent risk aversion implies that in the presence of a dependent background risk, a small risk affects the risk premium significantly. This means that in the presence of a dependent background risk, the risk premium associated with a small risk is mainly explained by $O(k)$, or by the first-degree conditional risk aversion.

Definition 3.1 is useful for deriving a first-order approximation of $\pi_{cd}(k)$.

Lemma 4.2

$$\pi_{cd}(k) = -k\frac{\int_{-\infty}^{\infty} ED(y)u_{12}(w, y)F_y(y)dy}{Eu_1(w, \tilde{y})} + O(k^2). \tag{8}$$

Proof See Appendix.

Lemma 4.2 shows the general measure for first-order conditional risk aversion. The first term involves two important concepts: u_{12}, the cross-derivative of the utility function, and $ED(y)$, the expectation dependence between the two risks. The sign of u_{12} indicates how this first concept acts on utility. Eeckhoudt et al. (2007) provide a context-free interpretation of the sign of u_{12}. They show that $u_{12} < 0$ is necessary and sufficient for "correlation aversion," meaning that a higher level of the background variable mitigates the detrimental effect of a reduction in wealth. Equation (8) also captures the dependence between the two risks. The sign of expectation dependence indicates whether the movements on background risk tend to reinforce the movements of $\tilde{y}$ on wealth (positive expectation dependence) or to counteract them (negative expectation dependence). As already discussed, $O(k^2)$ in (8) is negligible for small risks. Lemma 4.2 allows a quantitative treatment of the direction and size of the effect of expectation dependence on first order risk aversion. To clarify this, consider the following cases: (1) If the agents are correlation-neutral ($u_{12} = 0$) or the background risk is independent ($ED(y) = 0$), then the agents' attitude towards risk is only second-order conditional dependent risk aversion; and (2) If $u_{12} < 0$ and $ED(y) > 0$ ($ED(y) < 0$), then the agents' attitude towards risk is first-order conditional dependent, and their marginal risk premium for a small risk is positive (negative) (i.e., $\lim_{k \to 0^+} \pi'_{cd}(k) > (<) 0$).

¿From Lemma (4.2) and Equation (5), we obtain[2]

Proposition 4.3 *(i) If $\tilde{\varepsilon}$ is positive expectation dependent on $\tilde{y}$ and $u_{12} < 0$, then the agent's attitude towards risk is first-order conditional dependent risk aversion and $\pi_{cd}(k) - \pi_c(k) = |O(k)|$;*

(ii) If $\tilde{\varepsilon}$ is negative expectation dependent on $\tilde{y}$ and $u_{12} > 0$, then the agent's attitude towards risk is first-order conditional dependent risk aversion and $\pi_{cd}(k) - \pi_c(k) = |O(k)|$;

(iii) If $\tilde{\varepsilon}$ is positive expectation dependent on $\tilde{y}$ and $u_{12} > 0$, then the agent's attitude towards risk is first-order conditional dependent risk aversion and $\pi_{cd}(k) - \pi_c(k) = -|O(k)|$;

(iv) If $\tilde{\varepsilon}$ is negative expectation dependent on $\tilde{y}$ and $u_{12} < 0$, then the agent's attitude towards risk is first-order conditional dependent risk aversion and $\pi_{cd}(k) - \pi_c(k) = -|O(k)|$.

[2]Dionne and Li (2011) show that the more information we have about the sign of higher cross derivatives of the utility, the weaker the dependence conditions on distribution required to guarantee first-order conditional dependent risk aversion.

The intuition behind Proposition 4.3 is that in the absence of a dependent background risk, differentiable EU is only second order because the derivative is taken around the certainty line ($k = 0$, meaning no risk). In small neighborhoods differentiable functions behave like linear functions (in the present context, expected value, hence risk neutrality, as in (4)). But suppose that risk $\tilde{\varepsilon}$ is added to risk $\tilde{y}$ and the two are not independent. For example, suppose that $\tilde{y} = (-x, H; x, T)$ and $k\tilde{\varepsilon} = (-k, H; k, T)$; H and T are the same for $\tilde{y}$ and $k\tilde{\varepsilon}$ and represent two states of nature. From hereon, when we take the derivative of $\pi(k)$ with respect to k, at $k = 0$, we take derivatives of the utility function at two different points: $w - x$ and $w + x$. Because these derivatives are typically different under risk aversion, the derivative with respect to k will not be zero, but rather a function of the difference between $U'(w - x)$ and $U'(w + x)$ at $k = 0$, when the two risks are additive.

Consider a truncated standardized bivariate normal distribution with $h_1 < \tilde{\varepsilon} < +\infty$, $k_1 < \tilde{y} < +\infty$, and unconditional correlation coefficient ρ. This truncated standardized bivariate normal distribution is not elliptical. From Ang and Chen (2002, p. 488), we know that $\rho > 0$ cannot guarantee positive expectation dependence (for simulations on truncated standardized bivariate normal distribution, we refer to Dionne, Li, and Okou, 2012). The sign of covariance therefore cannot predict the sign of $\pi_{cd}(k)$. As Proposition 4.3 shows, the sign of expectation dependence can predict first-order risk aversion.

4.2 Two special cases and two examples

We consider two special cases to illustrate Proposition 4.3.

Case 1. Consider an additive background risk with $u(x, y) = U(x + y)$. Here, x may be the random wealth of an agent and y may be a random labor income risk that cannot be insured. Because $u_{12} < 0 \Leftrightarrow U'' < 0$, parts (i) and (iv) of Proposition 4.3 imply that if the agent is risk averse and $\tilde{\varepsilon}$ is positive (negative) expectation dependent on the background risk $\tilde{y}$, then the agent's attitude towards risk is first-order conditional dependent risk aversion and $\pi_{cd}(k) > (<) \pi_c(k)$. Another possible interpretation is that $\tilde{x}$ is the aggregate per capita dividend risk and $\tilde{y} = \tilde{W}_i$ is the idiosyncratic risk, where $\tilde{x} = \tilde{C}$ and $\tilde{W}_i$ are defined as in Examples 1 and 2. Then the agent is first-order conditional dependent risk-averse if $U'' \leq 0$.

Case 2. Consider a multiplicative background risk with $u(x, y) = U(xy)$. Here, x may be the random wealth of an agent and $\tilde{y} = (1 + \widetilde{r})$ where $\widetilde{r}$ is the random interest rate risk that cannot be hedged. Proposition 4.3 implies that (i) if relative risk aversion $-xy\frac{U''(xy)}{U'(xy)} > 1$ and $\tilde{\varepsilon}$ is positive (negative) expectation dependent on the background risk $\tilde{y}$, then the agent's attitude towards risk is first-order conditional dependent risk aversion and $\pi_{cd}(k) > (<)\pi_c(k)$; and (ii) if relative risk aversion $-xy\frac{U''(xy)}{U'(xy)} < 1$ and $\tilde{\varepsilon}$ is positive (negative) expectation dependent on the background risk $\tilde{y}$, then the agent's attitude towards risk is first-order conditional dependent risk aversion and $\pi_{cd}(k) < (>)\pi_c(k)$.

Consider bivariate log-normal random variables $(\tilde{\varepsilon}, \tilde{y})$ with joint probability distribution $F(\varepsilon, y)$ such that

$$\begin{pmatrix} \log(\tilde{\varepsilon}) \\ \log(\tilde{y}) \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix} \right), \tag{9}$$

where indexes 1 and 2 are used for $\log(\tilde{\varepsilon})$ and $\log(\tilde{y})$ respectively. We know from Lien (1985, pp. 244–245) that

$$ED(y) = \exp(\mu_1 + \frac{\sigma_1^2}{2})\frac{\Psi(\frac{\log(y)-\mu_2-\sigma_{12}}{\sigma_2})}{\Psi(\frac{\log(y)-\mu_2}{\sigma_2})}, \tag{10}$$

where $\Psi(z)$ is the cumulative distribution function of a standardized normal random variable evaluated at z.

Assume that $u(w+\tilde{x}+\tilde{y}) = \frac{[(w+\tilde{x})\tilde{y}]^{1-\gamma}}{1-\gamma}$ and $\tilde{x} = \overline{x} + k\tilde{\varepsilon}$ with $E(\varepsilon) = 0$. w is non-random initial wealth. Figure 1 presents different values of π_{cd} in function of k and γ when $\sigma_1 = 1, \sigma_2 = 10, \sigma_{12} = Cov(\log(\varepsilon), \log(y)) = \rho\sigma_1\sigma_2 = -1$, and $O(k^2) = 0$. We observe that π_{cd} is linear in k and decreases significantly with γ. This example corresponds to (iv) in Proposition 4.3. Since $\tilde{x}$ and $\tilde{y}$ are negatively correlated, there is a natural hedging between the two random variables and a negative premium. The premium decreases with k for a given γ because the size of risk increases with k and the natural hedging becomes more important. More risk averse individuals benefit more from the natural hedging and the effect is significant.

Example 3 The first example concerns a coward gambler. A coward gambler does not take a sufficiently small bet when offered two-to-one odds and permitted to choose a side (Samuelson, 1963). The assumption that people are cowards gamblers has some clear implications to financial markets (Segal and Spivak, 1990), as we will see in Example 4. The following proposition yields the conditions for people to be cowards (or not) in the presence of a background risk.

Proposition 4.4 *(i) Let $E[\tilde{\varepsilon}] > 0$. If the decision maker's attitude towards risk is second-order conditional dependent risk aversion, then for a sufficiently small $k > 0$, $Eu(w + k\tilde{\varepsilon}, \tilde{y}) > Eu(w, \tilde{y})$.*

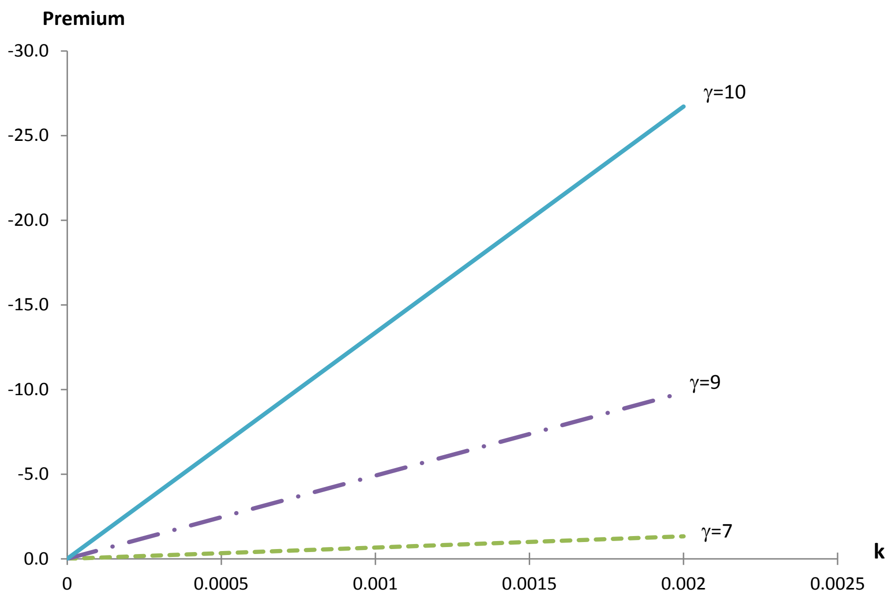

Figure 1
Conditional dependent risk premium (π_{cd} when $O\left(k^2\right) \approx 0$)
in function of risk size (k) and CRRA parameters (γ)

(ii) Let $E[\tilde{\varepsilon}] > 0$. If the individual's attitude towards risk is first-order conditional dependent risk aversion and $\pi_{cd}(k) - \pi_c(k) = |O(k)|$, then for a sufficiently small $k > 0$, $Eu(w, \tilde{y}) > Eu(w + k\tilde{\varepsilon}, \tilde{y})$, if $E[\tilde{\varepsilon}]$ is small enough.

Proof See Appendix.

Proposition 4.4 is also important for the next puzzle.

Example 4 Consider now a portfolio problem with two dependent risks: the payoff of the risky asset and the wage of the agent (for example an IBM employee considering buying IBM stock). The agent has guaranteed wealth w at the beginning of the period. The risk-free return over the period is r_f. The return of the risky asset over the period is a random variable $\tilde{r}$. The agent's problem is to determine how much to invest in the risky asset. Let $w - \alpha$ be the amount invested in the risk-free asset and α the amount invested in the risky asset. The value of wealth at the end of the period may be written as

$$\tilde{w} = (w - \alpha)(1 + r_f) + \alpha(1 + \tilde{r}) = w(1 + r_f) + \alpha(\tilde{r} - r_f) = w_0 + \alpha\tilde{x}, \tag{11}$$

where $w_0 = w(1 + r_f)$ is future wealth obtained with the risk-free asset and $\tilde{x} = \tilde{r} - r_f$. The agent's expected utility is

$$V(\alpha) = Eu(\tilde{w}, \tilde{y}) = Eu(w_0 + \alpha\tilde{x}, \tilde{y}). \tag{12}$$

Suppose $u_{11} \leq 0$. Because V is concave in α, the agent will invest a positive amount in the risky asset if and only if $Eu(w_0 + \alpha\tilde{x}, \tilde{y}) > Eu(w_0, \tilde{y})$ for small $\alpha > 0$. From part (i) of Proposition 4.4, we know that if the decision maker's attitude towards risk is second-order conditional dependent risk aversion, then the agent will invest a positive amount in the risky asset if it has a positive risk premium ($E(\tilde{r}) > r_f$). This replicates Arrow's (1974) famous result. However, part (ii) of Proposition 4.4 tells us that this conclusion fails when the decision maker is first-order conditional dependent risk-averse. In this case, we get the much more plausible result that if the risk premium is positive but sufficiently small, a correlation-averse decision maker will not invest in the risky asset if it is positive expectation dependent on the background risk. This last result is related to the financial market participation puzzle (Berbaris et al., 2006). It is well documented that the expected utility model (without a background risk) cannot explain the rejection of a small and actuarially favorable gamble. EU decision makers with second-order risk aversion will always invest part of their wealth in a risky asset with a positive risk premium, while first-order risk averse investors may not invest in this risky asset (Segal and Spivak, 1990). Part (ii) of Proposition 4.4 shows that this last result is compatible with an EU decision maker in the presence of a dependent background risk.

5 Applications

In this section, we illustrate the applicability of our results to two problems (puzzles) analyzed in the literature. In particular, we demonstrate the practical relevance of the distinction between first- and second-order risk aversion. In these applications, the background risk is interpreted as an idiosyncratic risk. For example, in a permanent income shock macroeconomic environment, full insurance of background risk is not possible (see, for example, Constantinides and Duffie, 1996). Therefore, at equilibrium there is no trading between consumers to eliminate the idiosyncratic risk. Consumption shocks translate into idiosyncratic income shocks and background risk remains significant.

5.1 The equity premium puzzle

The equity premium puzzle illustrates that second-order risk aversion is not sufficient to explain asset prices. Epstein and Zin (1990) solve the puzzle with rank-dependent expected utility, which exhibits first-order risk aversion. Another explanation involves incomplete market models with idiosyncratic risk. Constantinides and Duffie (1996) show how the introduction of heterogeneity in the form of uninsurable, persistent and heteroskedastic labor income shocks may resolve the puzzle in theory. Krueger and Lustig (2010) present five conditions to affirm that market incompleteness does not increase the risk premium. One of these conditions is the independence between idiosyncratic risk and aggregate shocks. In this subsection, we show how dependent risks increase the risk premium in the framework of expected utility.

We concentrate our analysis on the independence assumption by considering the standard representative agent model with two dependent risks. We use the static version of Weil's (1992) model proposed by Gollier and Schlesinger (2002) to illustrate our main result. Our model with dependent risks is a particular case of the intertemporal setting of Krueger and Lustig (2010), but adequately illustrates the effect of a dependent idiosyncratic risk on asset price. Note that our measure of risk dependence is very general and that our result holds for (almost) all concave expected utility functions, including constant relative risk aversion (CRRA) and constant absolute risk aversion (CARA) preferences.

We consider a static Lucas (1978) tree economy which consists of individuals, all of whom may be portrayed, ex ante, by a representative agent. The economy is competitive in that individuals maximize expected utility with prices taken as given. Initial wealth consists of one unit of the risky asset plus an allocation of a risk-free asset. We assume that the risk-free rate is zero. We denote w as the value of wealth that is initially invested in the risk-free asset and define $\tilde{x}$ as the final payoff of the risky asset. Agents' preferences can be represented by a von Neumann-Morgenstern utility function u. Agents can adjust their portfolio by buying and selling the two assets, and consume all their wealth at the end of the period. Let P represent the price of the risky asset and β denote the demand for additional units of $\tilde{x}$. In the absence of a background risk, we assume that the agent faces the following optimization program:

$$\beta^* \in \arg\max_\beta Eu(w + \tilde{x} + \beta(\tilde{x} - P), E\tilde{y}). \tag{13}$$

Gollier and Schlesinger (2002) show that the equilibrium asset price with an excess demand of zero ($\beta^* = 0$) is equal to

$$P^* = \frac{E[\tilde{x}u_1(w + \tilde{x}, E\tilde{y})]}{Eu_1(w + \tilde{x}, E\tilde{y})}. \tag{14}$$

One way to explain the equity premium puzzle in this theoretical framework is to recognize that there are other sources of risk on final wealth than the riskiness of asset returns. To capture the effects of these types of risks, we introduce a background risk $\tilde{y}$, which can not be fully insured. This yields the following optimization program:

$$\beta^{**} \in \arg\max_\beta Eu(w + \tilde{x} + \beta(\tilde{x} - P), \tilde{y}), \tag{15}$$

and modified equilibrium asset price:

$$P^{**} = \frac{E[\tilde{x}u_1(w + \tilde{x}, \tilde{y})]}{Eu_1(w + \tilde{x}, \tilde{y})}. \tag{16}$$

We want to compare P^{**} with P^*. Because $E(\tilde{x})$ is the same in both cases and the equity premium is $E(\tilde{x}) - P$ (see Gollier and Schlesinger, 2002, p. 756), this comparison is identical to comparing the equity premiums.

Suppose $u(x, y) = \frac{[(w+x)y]^{1-\gamma}}{1-\gamma}$, and $\tilde{x}$ and $\tilde{y}$ are independent. $\tilde{y}$ is multiplicative, as in Krueger and Lustig (2010) and Constantinides and Duffie (1996). Then $u_1 = (w + x)^{-\gamma} y^{1-\gamma}$ and

$$P^* = \frac{E\tilde{x}(w + \tilde{x})^{-\gamma}(E\tilde{y})^{1-\gamma}}{E(w + \tilde{x})^{-\gamma}(E\tilde{y})^{1-\gamma}} = \frac{E\tilde{x}(w + \tilde{x})^{-\gamma}}{E(w + \tilde{x})^{-\gamma}} \tag{17}$$

while

$$P^{**} = \frac{E\tilde{x}(w+\tilde{x})^{-\gamma}\tilde{y}^{1-\gamma}}{E(w+\tilde{x})^{-\gamma}\tilde{y}^{1-\gamma}} = \frac{E\tilde{x}(w+\tilde{x})^{-\gamma}E\tilde{y}^{1-\gamma}}{E(w+\tilde{x})^{-\gamma}E\tilde{y}^{1-\gamma}} = \frac{E\tilde{x}(w+\tilde{x})^{-\gamma}}{E(w+\tilde{x})^{-\gamma}}. \tag{18}$$

We verify that $P^* = P^{**}$. We obtain the irrelevance result for the CRRA utility function and independent multiplicative risks. This conclusion meets Krueger and Lustig (2010, pp. 9-10) when all other conditions are verified. We can also obtain the irrelevance result of Weil (1992) with independent additive risks.

We want to find the condition under which idiosyncratic risk is a potential solution to the equity premium puzzle. We suppose that $\tilde{x} = \bar{x} + k\tilde{\varepsilon}$ with $E\tilde{\varepsilon} = 0$ to maintain the assumption of a small risk. We obtain the following result.

Proposition 5.1

$$P^{**} - P^* = \tag{19}$$
$$\frac{u_1(w+\bar{x}, E\tilde{y})\int_{-\infty}^{\infty} ED(y)u_{12}(w+\bar{x}, y)F_y(y)dy}{[Eu_1(w+\bar{x},\tilde{y}) + k\int_{-\infty}^{\infty} ED(y)u_{112}(w+\bar{x},y)F_y(y)dy + O(k^2)][u_1(w+\bar{x}, E\tilde{y}) + O(k^2)]}k$$
$$+O(k^2).$$

Proof See Appendix.

For small risk, we have the following conclusions. (i) If $\tilde{\varepsilon}$ and $\tilde{y}$ are independent, then $ED(y) = 0$ for all y, and $P^{**} - P^* = O(k^2)$. Again, this difference is very small and even nil for many utility functions used in the financial literature, as shown above; (ii) If $\tilde{\varepsilon}$ is positive expectation dependent on $\tilde{y}$ and $u_{12} < 0$, then $P^{**} - P^* = -|O(k)|$; and (iii) If $\tilde{\varepsilon}$ is negative expectation dependent on $\tilde{y}$ and $u_{12} < 0$, then $P^{**} - P^* = |O(k)|$.

¿From (8), we can rewrite $P^{**} - P^*$ as

$$P^{**} - P^* = -B\pi_{cd}(k) + O(k^2), \tag{20}$$

where $\pi_{cd}(k) = -k\frac{\int_{-\infty}^{\infty} ED(y)u_{12}(w+\bar{x},y)F_y(y)dy}{Eu_1(w+\bar{x},\tilde{y})} + O(k^2)$ as in (8), and

$$B = \tag{21}$$
$$\frac{u_1(w+\bar{x}, E\tilde{y})Eu_1(w+\bar{x},\tilde{y})}{[Eu_1(w+\bar{x},\tilde{y}) + k\int_{-\infty}^{\infty} ED(y)u_{112}(w+\bar{x},y)F_y(y)dy + O(k^2)][u_1(w+\bar{x}, E\tilde{y}) + O(k^2)]}$$

is a positive constant if k is small enough. This is true because when k is small, the sign of u_{112} does not matter. Therefore, (20) indicates that when the risk is small, the equity premium is determined by the order of conditional dependent risk aversion ($\pi_{cd}(k)$). Our result shows that independent background risk can only generate a second-order effect, whereas an expectation dependent background risk can generate a first-order effect, and hence offers a better understanding of the effect of incomplete markets on asset prices

Another way to interpret our result is the following. Suppose that

$$u(w + \tilde{x} + \beta(\tilde{x} - P), \tilde{y}) = U\left(w + \tilde{x} + \beta\left(\tilde{x} - P\right) + \tilde{W}_i\right),$$

where $\tilde{W}_i = (aI_{\{\tilde{x} \leq x^*\}} + bI_{\{\tilde{x} > x^*\}})g\tilde{u}_i$ or $e^{\tilde{\delta}_i}\tilde{x}$ is a dependent idiosyncratic risk, a, b, g, $\tilde{u}_i$ and $\tilde{\delta}_i$ are defined in Examples 1 and 2. $\tilde{W}_i$ is the labor income of agent i and $\tilde{x}$ is the outcome of the risky asset. Then $\tilde{x}$ can also be positive expectation dependent on $\tilde{W}_i$ when aggregate per capita consumption $\tilde{C}$ is positive expectation dependent on $\tilde{W}_i$.[3] Proposition 5.1 states that the asset price in an economy with a dependent idiosyncratic labor income risk $\tilde{W}_i$ should be significantly lower than the asset price in an economy with an independent idiosyncratic labor income risk.

In particular, (20) states that with a positive π_{cd}, we get a negative price difference $(P^{**} - P^*)$, which means a higher risk premium in the economy with a dependent (and persistent) idiosyncratic labor income risk that cannot be hedged. We can explain this result as follows: because $\tilde{x}$ and $\tilde{W}_i$ are positive expectation dependent, and the risk averse agent cannot hedge the simultaneous downside (upside) evolution of $(\tilde{x}, \tilde{W}_i)$, then the agent requires a higher risk premium for holding $\tilde{x}$ with $\tilde{W}_i$ than for holding $\tilde{x}$ with $E\tilde{W}_i$.

5.2 The welfare cost of business cycles

We now consider how eliminating all consumption variability affects welfare. Consider a representative consumer, endowed with the stochastic per capita consumption stream $\tilde{c} = \bar{c} + k\tilde{\varepsilon}$. Preferences over such consumption are assumed to be

$$Eu(\tilde{c}, \tilde{y}), \tag{22}$$

[3]Suppose $\tilde{x} = a\tilde{C} + \tilde{b}$, $a > 0$ and $\tilde{b}$ is a random variable independent of $\tilde{C}$, then $\tilde{x}$ is expectation dependent on $\tilde{W}_i$ if C is expectation dependent on $\tilde{W}_i$. The proof is similar to that in Proposition 3.3. With more general relationships between $\tilde{x}$ and $\tilde{C}$ maters are more complicated and the conclusion remains on empirical tests, as for the covariance between aggregate consumption and security returns discussed by Constantinides and Duffie (1996).

where $\tilde{y}$ is a background risk.

A risk-averse consumer would prefer a deterministic consumption to a risky one with the same mean under certain conditions. As in Lucas (1987), we quantify the utility difference between risk with compensation and certainty by multiplying the risky consumption by the constant factor $1+\lambda$. We choose a λ such that the household is indifferent between the deterministic consumption and the compensated risky one. In other words, λ^* solves

$$Eu((1+\lambda^*)\tilde{c},\tilde{y}) = Eu(E\tilde{c},\tilde{y}).$$

Lucas (1987; 2003) defines the compensation parameter λ^* as the welfare gain (or welfare loss) from eliminating consumption risk. When $\tilde{y}$ is an insurable risk ($\tilde{y} \equiv E\tilde{y}$), Lucas (1987; 2003) argues that the welfare costs of business cycles are likely to be very small, and that the potential gains from counter-cyclical stabilization policy are negligible. Therefore governments should look for ways to attain higher growth rates rather than for economic policies to reduce fluctuation in consumption. However, this conclusion conflicts with the actual practice of short-term economic policies in many countries. Epstein-Zin recursive utility and first-order risk aversion variations model yields bigger numbers (Dolmas, 1998; Epaulard and Pommeret, 2003). As we will see, incomplete market models can also offer higher numbers. Our result shows that dependent background risk appears to be a key factor.

We now apply our results to this issue. Because for $\forall\lambda$, we have

$$\begin{aligned} Eu((1+\lambda)\tilde{c},\tilde{y}) &= Eu((1+\lambda)\bar{c}+(1+\lambda)k\tilde{\varepsilon},\tilde{y}) \qquad (23)\\ &= Eu((1+\lambda)\bar{c}+(1+\lambda)(kE\tilde{\varepsilon}-\pi_{cd}(k)),\tilde{y}) \\ &= Eu(E\tilde{c}+\lambda\bar{c}+\lambda kE\tilde{\varepsilon}-(1+\lambda)\pi_{cd}(k),\tilde{y}), \end{aligned}$$

hence $\lambda^* = \frac{\pi_{cd}(k)}{\bar{c}+kE\tilde{\varepsilon}-\pi_{cd}(k)}$ and we obtain that:

Proposition 5.2 *Suppose the representative consumer is risk-averse and k is small (i.e. $\bar{c} > O(k)$).*

(i) If $\tilde{c}$ and $\tilde{y}$ are independent, then the consumer's attitude towards risk is second-order conditional dependent risk aversion and $\lambda^* = O(k^2)$;

(ii) If $\tilde{c}$ is positive expectation dependent on $\tilde{y}$, then the consumer's attitude towards risk is first-order conditional dependent risk aversion and $\lambda^* = O(k)$.

Proposition 5.2 states that when the risk associated with consumption is small, we obtain the following: (i) If the consumption risk and the background risk are independent, then the welfare costs of business cycles is very small, and therefore the potential gains from counter-cyclical stabilization policy are negligible; (ii) If consumption risk is positive expectation dependent on the background risk, then the welfare costs of business cycles can be large, and the potential gains from a counter-cyclical stabilization policy may be significant.

To obtain a more intuitive interpretation of our result, we can define $u(c, y) = U(c_i) = U(c + y_i)$, where $\tilde{c} = \tilde{C}$ is per capita consumption and $\tilde{y}_i = \tilde{W}_i$ is, in this case, an idiosyncratic shock on consumption that cannot be diversified (De Santis, 2007). The dependence between $\tilde{C}$ and $\tilde{W}_i$ can be defined as in Examples 1 and 2. $\tilde{C}$ is then positive expectation dependent on $\tilde{W}_i$. Proposition 5.2 shows that in the presence of a permanent idiosyncratic shock on consumption, the potential gains from a counter-cyclical stabilization policy will be more significant than in a situation with a less permanent shock that can be insurable. De Santis (2007) obtained a similar result with a different model. Here the main effect is first-order instead of second-order.

6 Link with rank-dependent expected utility theory

Because Epstein and Zin (1990) extended rank-dependent expected utility (RDEU) to intertemporal utility, RDEU has become an important source of first-order risk aversion in the macroeconomics literature. In this section, we show the connection between our results and RDEU for generating first-order risk aversion.

In RDEU (Quiggin 1982; Segal 1990), there is a strictly increasing and continuous function $g : [0, 1] \to [0, 1]$, such that

$$RDEU(\tilde{x}) = \int_m^M U(x)dg(F_x(x)). \tag{24}$$

Because

$$\begin{aligned} RDEU(\tilde{x}) &= \int_m^M U(x)dg(F_x(x)) \qquad (25) \\ &= \int_m^M U(x)g'(F_x(x))dF_x(x) \\ &= E[U(\tilde{x})g'(F_x(\tilde{x}))] \\ &= EV(\tilde{x}, \tilde{y}), \end{aligned}$$

where $V(x, y) = U(x)y$ and $\tilde{y} = g'(F_x(\tilde{x}))$, we can view the rank ordering of outcomes as a background risk $\tilde{y}$ in the framework of expected utility. The concavity (convexity) of g is a necessary and sufficient condition for risk aversion (risk loving) in RDEU model, because the probabilities of the worst (best) consequences are over-weighted compared to their untransformed probabilities (Chew et al., 1987; Yaari, 1987). Because $g'' \leq (\geq) 0 \Rightarrow cov(\tilde{x}, m(\tilde{y})) \leq (\geq) 0$ for every increasing function m, $\tilde{x}$ is negative (positive) expectation dependent on $\tilde{y}$ if g is concave (convex). We can say that when the agent is risk-averse (risk-loving) in the RDEU model, $\tilde{y}$ is a negative (positive) expectation dependent background risk.

We define the risk premium for RDEU, $\pi^{RDEU}(k)$, as

$$RDEU(Ek\tilde{x} - \pi^{RDEU}(k)) = RDEU(k\tilde{x}). \tag{26}$$

(25) implies $\pi^{RDEU}(k) = \pi_{cd}(k)$, where $\pi_{cd}(k)$ is defined as $EV(w + Ek\tilde{x} - \pi_{cd}(k), \tilde{y}) = EV(w + k\tilde{x}, \tilde{y})$.

From Proposition 4.3 and the fact that $V_{12} \geq 0$, we obtain $\pi^{RDEU}(k) = \pi_{cd}(k) = |O(k)|$ (or $-|O(k)|$) if g is concave (convex). This meets Segal and Spivak's result (1990, Proposition 4). Therefore, we identify a formal connection between dependent background risk and RDEU to generate first-order risk aversion. Because RDEU is the source of first-order risk aversion in the Epstein-Zin model, and we can represent RDEU as expected utility with dependent background risk, our result provides a new interpretation of the Epstein-Zin model via dependent background risk in the framework of expected utility.

7 Conclusion

This paper shows that differentiable expected utility can be compatible with first-order risk behavior if there exists a risk that can be eliminated and a background risk that cannot be eliminated (such as uncertain labor income), but these two risks are dependent. We have presented a set of applications and examples to illustrate that the practical relevance of the distinction between first- and second-order risk aversion is significant. We offer a better understanding of some economic and finance real world situations, and how they relate to the non-standard preference models.

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9 Appendix: Proofs

9.1 Proof of Proposition 3.3

Suppose $h(\varepsilon, u)$ is increasing in ε. By law of total covariance[4], we have

$$\begin{aligned}
& Cov(\tilde{\varepsilon}, \gamma(\tilde{y})) \qquad (27)\\
= \; & Cov(\tilde{\varepsilon}, \gamma(h(\tilde{\varepsilon}, \tilde{u})) \\
= \; & E[Cov(\tilde{\varepsilon}, \gamma(h(\tilde{\varepsilon}, \tilde{u})|\tilde{u})] + Cov((E\tilde{\varepsilon}|\tilde{u}), E(\gamma(h(\tilde{\varepsilon}, \tilde{u}))|\tilde{u})) \\
= \; & E[Cov(\tilde{\varepsilon}, \gamma(h(\tilde{\varepsilon}, \tilde{u})|\tilde{u})] + Cov(E\tilde{\varepsilon}, E(\gamma(h(\tilde{\varepsilon}, \tilde{u}))|\tilde{u})) \\
= \; & E[Cov(\tilde{\varepsilon}, \gamma(h(\tilde{\varepsilon}, \tilde{u})|\tilde{u})] \geq 0
\end{aligned}$$

for every increasing γ. Therefore, $\tilde{\varepsilon}$ is positive expectation dependent on $\tilde{y}$.

Suppose $h(\varepsilon, u)$ is decreasing in ε. By a similar approach, we can show that $\tilde{\varepsilon}$ is negative expectation dependent on $\tilde{y}$.

9.2 Proof of Lemma 4.2

From the definition of $\pi_{cd}(k)$, we know that

$$Eu(w + Ek\tilde{\varepsilon} - \pi_{cd}(k), \tilde{y}) = Eu(w + k\tilde{\varepsilon}, \tilde{y}). \qquad (28)$$

Differentiating with respect to k yields

$$\pi'_{cd}(k) = \frac{E\tilde{\varepsilon}Eu(w + Ek\tilde{\varepsilon} - \pi_{cd}(k), \tilde{y}) - E[\tilde{\varepsilon}u_1(w + k\tilde{\varepsilon}, \tilde{y})]}{Eu_1(w - \pi_{cd}(k), \tilde{y})}. \qquad (29)$$

[4] $Cov(\tilde{X}, \tilde{Y}) = E[Cov(\tilde{X}, \tilde{Y}|\tilde{Z})] + Cov(E(\tilde{X}|\tilde{Z}), E(\tilde{Y}|\tilde{Z}))$.

Because $\pi_{cd}(0) = 0$, we have

$$\pi'_{cd}(0) = \frac{E\tilde{\varepsilon}Eu_1(w,\tilde{y}) - E[\tilde{\varepsilon}u_1(w,\tilde{y})]}{Eu_1(w,\tilde{y})}. \tag{30}$$

Note that

$$E[\tilde{\varepsilon}u_1(w,\tilde{y})] = E\tilde{\varepsilon}Eu_1(w,\tilde{y}) + Cov(\tilde{\varepsilon}, u_1(w,\tilde{y})) \tag{31}$$

and the covariance can always be written as (see Cuadras, 2002, Theorem 1)

$$Cov(\tilde{\varepsilon}, u_1(w,\tilde{y})) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[F(\varepsilon,y) - F_\varepsilon(\varepsilon)F_y(y)]d\varepsilon du_1(w,y). \tag{32}$$

Because from Lemma 1 in Tesfatsion (1976)

$$\int_{-\infty}^{\infty}[F_\varepsilon(\varepsilon|\tilde{y} \le y) - F_\varepsilon(\varepsilon)]d\varepsilon = E\tilde{\varepsilon} - E(\tilde{\varepsilon}|\tilde{y} \le y), \tag{33}$$

hence, by straightforward manipulations, we find

$$\begin{aligned} Cov(\tilde{\varepsilon}, u_1(w,\tilde{y})) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[F(\varepsilon,y) - F_\varepsilon(\varepsilon)F_y(y)]u_{12}(w_0,y)d\varepsilon dy \qquad (34) \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[F_\varepsilon(\varepsilon|\tilde{y} \le y) - F_\varepsilon(\varepsilon)]d\varepsilon F_y(y)u_{12}(w,y)dy \\ &= \int_{-\infty}^{\infty}[E\tilde{\varepsilon} - E(\tilde{\varepsilon}|\tilde{y} \le y)]F_y(y)u_{12}(w,y)dy \qquad (by\ (33)) \\ &= \int_{-\infty}^{\infty} ED(y)u_{12}(w,y)F_y(y)dy. \end{aligned}$$

Finally, we get

$$\pi'_{cd}(0) = -\frac{\int_{-\infty}^{\infty} ED(y)u_{12}(w,y)F_y(y)dy}{Eu_1(w,\tilde{y})}. \tag{35}$$

Using a Taylor expansion of π around $k = 0$, we obtain that

$$\pi_{cd}(k) = \pi_{cd}(0) + \pi'_{cd}(0)k + O(k^2) = -k\frac{\int_{-\infty}^{\infty} ED(y)u_{12}(w,y)F_y(y)dy}{Eu_1(w,\tilde{y})} + O(k^2). \tag{36}$$

Q.E.D.

9.3 Proof of Proposition 4.4

(i) Suppose the decision maker's attitude towards risk is second-order conditional dependent risk aversion, then $\pi_{cd}(k) - \pi_c(k) = O(k^2)$. From (7), we have

$$\begin{aligned} & Eu(w + k\tilde{\varepsilon}, \tilde{y}) \qquad (37) \\ = \; & Eu(w + Ek\tilde{\varepsilon} - \pi_{cd}(k), \tilde{y}) \\ = \; & Eu(w + kE\tilde{\varepsilon} - O(k^2) - \pi_c(k), \tilde{y}) \\ = \; & Eu(w + kE\tilde{\varepsilon} - O(k^2), \tilde{y}) \quad by\ (5) \\ > \; & Eu(w, \tilde{y}) \quad for\ a\ sufficiently\ small\ k > 0. \end{aligned}$$

(ii) Suppose the decision maker's attitude towards risk is first-order conditional dependent risk aversion and $\pi_{cd}(k) - \pi_c(k) = |O(k)|$. ¿From (7), we have

$$\begin{aligned} & Eu(w + k\tilde{\varepsilon}, \tilde{y}) \qquad (38) \\ = \; & Eu(w + Ek\tilde{\varepsilon} - \pi_{cd}(k), \tilde{y}) \\ = \; & Eu(w + kE\tilde{\varepsilon} - |O(k)| - \pi_c(k), \tilde{y}) \\ = \; & Eu(w + kE\tilde{\varepsilon} - |O(k)| - O(k^2), \tilde{y}) \quad by\ (5) \\ < \; & Eu(w, \tilde{y}) \quad for\ a\ sufficiently\ small\ k > 0\ and\ a\ sufficiently\ small\ E\tilde{\varepsilon}. \end{aligned}$$

Q.E.D.

9.4 Proof of Proposition 5.1

Using a Taylor expansion of π around $k = 0$, we obtain that

$$\begin{aligned} & u_1(w + \bar{x} + k\tilde{\varepsilon}, E\tilde{y}) \qquad (39) \\ = \; & u_1(w + \bar{x}, E\tilde{y}) + \tilde{\varepsilon} u_{11}(w + \bar{x}, E\tilde{y})k + O(k^2), \end{aligned}$$

$$\begin{aligned} & (\bar{x} + k\tilde{\varepsilon})u_1(w + \bar{x} + k\tilde{\varepsilon}, E\tilde{y}) \qquad (40) \\ = \; & \bar{x}u_1(w + \bar{x}, E\tilde{y}) + [\tilde{\varepsilon}u_1(w + \bar{x}, E\tilde{y}) + (\bar{x} + k\tilde{\varepsilon})\tilde{\varepsilon}u_{11}(w + \bar{x}, E\tilde{y})]k + O(k^2) \\ = \; & \bar{x}u_1(w + \bar{x}, E\tilde{y}) + \tilde{\varepsilon}u_1(w + \bar{x}, E\tilde{y})k + \bar{x}\tilde{\varepsilon}u_{11}(w + \bar{x}, E\tilde{y})k + \tilde{\varepsilon}^2 u_{11}(w + \bar{x}, E\tilde{y})k^2 + O(k^2) \\ = \; & \bar{x}u_1(w + \bar{x}, E\tilde{y}) + \tilde{\varepsilon}u_1(w + \bar{x}, E\tilde{y})k + \bar{x}\tilde{\varepsilon}u_{11}(w + \bar{x}, E\tilde{y})k + O(k^2), \end{aligned}$$

$$\begin{aligned} & u_1(w + \bar{x} + k\tilde{\varepsilon}, \tilde{y}) \qquad (41) \\ = \; & u_1(w + \bar{x}, \tilde{y}) + \tilde{\varepsilon}u_{11}(w + \bar{x}, \tilde{y})k + O(k^2) \end{aligned}$$

and

$$\begin{aligned} & (\bar{x} + k\tilde{\varepsilon})u_1(w + \bar{x} + k\tilde{\varepsilon}, \tilde{y}) \qquad (42) \\ = \; & \bar{x}u_1(w + \bar{x}, \tilde{y}) + [\tilde{\varepsilon}u_1(w + \bar{x}, \tilde{y}) + (\bar{x} + k\tilde{\varepsilon})\tilde{\varepsilon}u_{11}(w + \bar{x}, \tilde{y})]k + O(k^2) \\ = \; & \bar{x}u_1(w + \bar{x}, \tilde{y}) + \tilde{\varepsilon}u_1(w + \bar{x}, \tilde{y})k + \bar{x}\tilde{\varepsilon}u_{11}(w + \bar{x}, \tilde{y})k + \tilde{\varepsilon}^2 u_{11}(w + \bar{x}, \tilde{y})k^2 + O(k^2) \\ = \; & \bar{x}u_1(w + \bar{x}, \tilde{y}) + \tilde{\varepsilon}u_1(w + \bar{x}, \tilde{y})k + \bar{x}\tilde{\varepsilon}u_{11}(w + \bar{x}, \tilde{y})k + O(k^2). \end{aligned}$$

From (39), we have

$$Eu_1(w + \bar{x} + k\tilde{\varepsilon}, E\tilde{y}) = u_1(w + \bar{x}, E\tilde{y}) + O(k^2). \qquad (43)$$

From (40), we have

$$E(\bar{x}+k\tilde{\varepsilon})u_1(w+\bar{x}+k\tilde{\varepsilon},E\tilde{y}) = \bar{x}u_1(w+\bar{x},E\tilde{y}) + O(k^2). \tag{44}$$

From (41), we have

$$\begin{aligned}
& Eu_1(w+\bar{x}+k\tilde{\varepsilon},\tilde{y}) && (45)\\
= \; & Eu_1(w+\bar{x},\tilde{y}) + E\tilde{\varepsilon}u_{11}(w+\bar{x},\tilde{y})k + O(k^2)\\
= \; & Eu_1(w+\bar{x},\tilde{y}) + cov(\tilde{\varepsilon},u_{11}(w+\bar{x},\tilde{y}))k + O(k^2)\\
= \; & Eu_1(w+\bar{x},\tilde{y}) + k\int_{-\infty}^{\infty} ED(y)u_{112}(w+\bar{x},y)F_y(y)dy + O(k^2) && (46)
\end{aligned}$$

From (42), we have

$$\begin{aligned}
& E(\bar{x}+k\tilde{\varepsilon})u_1(w+\bar{x}+k\tilde{\varepsilon},\tilde{y}) && (47)\\
= \; & \bar{x}Eu_1(w+\bar{x},\tilde{y}) + E\tilde{\varepsilon}u_1(w+\bar{x},\tilde{y})k + E\tilde{\varepsilon}u_{11}(w+\bar{x},\tilde{y})\bar{x}k + O(k^2)\\
= \; & \bar{x}Eu_1(w+\bar{x},\tilde{y}) + cov(\tilde{\varepsilon},u_1(w+\bar{x},\tilde{y}))k + cov(\tilde{\varepsilon},u_{11}(w+\bar{x},\tilde{y}))\bar{x}k + O(k^2)\\
= \; & \bar{x}Eu_1(w+\bar{x},\tilde{y}) + k\int_{-\infty}^{\infty} ED(y)u_{12}(w+\bar{x},y)F_y(y)dy + \bar{x}k\int_{-\infty}^{\infty} ED(y)u_{112}(w+\bar{x},y)F_y(y)dy\\
& +O(k^2)\\
= \; & \bar{x}Eu_1(w+\bar{x},\tilde{y}) + k\int_{-\infty}^{\infty} ED(y)[u_{12}(w+\bar{x},y)+\bar{x}u_{112}(w+\bar{x},y)]F_y(y)dy + O(k^2).
\end{aligned}$$

Therefore,

$$\begin{aligned}
& P^{**} - P^{*} && (48)\\
= \; & \frac{E[\tilde{x}u_1(w+\tilde{x},\tilde{y})]}{Eu_1(w+\tilde{x},\tilde{y})} - \frac{E[\tilde{x}u_1(w+\tilde{x},E\tilde{y})]}{Eu_1(w+\tilde{x},E\tilde{y})}\\
= \; & \frac{\bar{x}Eu_1(w+\bar{x},\tilde{y}) + k\int_{-\infty}^{\infty} ED(y)[u_{12}(w+\bar{x},y)+\bar{x}u_{112}(w+\bar{x},y)]F_y(y)dy + O(k^2)}{Eu_1(w+\bar{x},\tilde{y}) + k\int_{-\infty}^{\infty} ED(y)u_{112}(w+\bar{x},y)F_y(y)dy + O(k^2)}\\
& -\frac{\bar{x}u_1(w+\bar{x},E\tilde{y}) + O(k^2)}{u_1(w+\bar{x},E\tilde{y}) + O(k^2)}\\
= \; & \frac{u_1(w+\bar{x},E\tilde{y})\int_{-\infty}^{\infty} ED(y)u_{12}(w+\bar{x},y)F_y(y)dy}{[Eu_1(w+\bar{x},\tilde{y}) + k\int_{-\infty}^{\infty} ED(y)u_{112}(w+\bar{x},y)F_y(y)dy + O(k^2)][u_1(w+\bar{x},E\tilde{y}) + O(k^2)]}k\\
& +O(k^2).
\end{aligned}$$

Q.E.D.

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