Comparative Ross Risk Aversion in the Presence of Mean Dependent Risks

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Abstract

This paper studies comparative risk aversion between risk averse agents in the presence of a background risk. Our contribution differs from most of the literature in two respects. First, background risk does not need to be additive or multiplicative. Second, the two risks are not necessarily mean independent, and may be conditional expectation increasing or decreasing. We show that our order of cross Ross risk aversion is equivalent to the order of partial risk premium, while our index of decreasing cross Ross risk aversion is equivalent to decreasing partial risk premium. These results generalize the comparative risk aversion model developed by Ross for mean independent risks. Our theoretical results propose new insights into comparing the welfare costs of business cycles and are related to utility functions having the n-switch independence property. They can be applied to many other economic situations implying a background risk.

Key words: Comparative cross Ross risk aversion, dependent background risk, partial risk premium, decreasing cross Ross risk aversion, n-switch independence property.

JEL classification: D81.

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1 Introduction

Arrow (1965) and Pratt (1964) propose an important theorem stating that risk aversion comparisons using risk premia and measures of risk aversion always give the same result. Ross (1981) shows that when an agent faces more than one risky variable, Arrow-Pratt measures are not strong enough to support the plausible association between absolute risk aversion and the size of the risk premium. He proposes a stronger ordering called Ross risk aversion. Several studies extend Ross' results. Most papers generalize them to higher-orders of risk aversion for univariate utility functions (see Modica and Scarsini, 2005; Jindapon and Neilson, 2007; Li, 2009; Denuit and Eeckhoudt, 2010a). Liu and Meyer (2012) use a "concave risk aversion property" to measure Ross risk aversion and to characterize Ross definition of strongly more risk averse on bounded intervals. This paper extends this line of research to bivariate utility functions.

There is growing concern about risk attitudes of bivariate utility function in the literature (see Courbage, 2001; Bleichrodt et al., 2003; Eeckhoudt et al., 2007; Courbage and Rey, 2007; Menegatti, 2009 a,b; Denuit and Eeckhoudt, 2010b; Li, 2011; Denuit et al., 2011a). To our knowledge, these studies do not analyze comparative risk aversion. The first paper that looks at preservation of "more risk averse" with general multivariate preferences and background risk is Nachman (1982). However, in his setting, the background risk is independent. Pratt (1988) also considers the comparison of risk aversion with the presence of an independent background risk using a two-argument utility function.

We generalize the model of comparative risk aversion developed by Ross (1981). We introduce the notion of cross Ross risk aversion and show that more cross Ross risk aversion is associated with a higher partial risk premium in the presence of a conditional expectation increasing (or decreasing) background risk. Hence, we demonstrate that the index of cross Ross risk aversion is equivalent to the order of partial risk premium. We also propose the concept of decreasing cross Ross risk aversion and derive necessary and sufficient conditions for obtaining an equivalence between decreasing cross Ross risk aversion and decreasing partial risk premium in the presence of a conditional background risk. We apply this result to examine the effects of changes in wealth and financial background risk on the intensity of risk aversion, to compare the welfare cost of business cycles and to show the relationship between decreasing cross Ross risk aversion and the n-switch independence property (Abbas and Bell, 2011). Other potential applications are discussed.

Our paper is organized as follows. Sections 2 and 3 offer the necessary and sufficient conditions for comparing two agents' attitudes towards risk with different utility functions and the same agent's attitude at different wealth levels under a conditional expectation increasing background risk. Section 4 extend the results to conditional expectation decreasing background risk. Section 5 applies our results to financial background risks. We compare the welfare cost of business cycles in Section 6. Section 7 relates decreasing risk aversion with an n-switch independence property and Section 8 concludes the paper and proposes two other economic applications.

2 Comparative cross risk attitudes

We consider an economic agent whose preference for wealth, $\tilde{w}$ and a random variable, $\tilde{y}$, can be represented by a bivariate model of expected utility. We let $u(w, y)$ denote the utility function, and let $u_1(w, y)$ denote $\frac{\partial u}{\partial w}$ and $u_2(w, y)$ denote $\frac{\partial u}{\partial y}$. We follow the same subscript convention for the derivatives $u_{11}(w, y)$, $u_{12}(w, y)$ and so on, and assume that the partial derivatives required for any definition all exist and are continuous.

Pratt (1990) and Chalfant and Finkelshtain (1993) introduce the following definition of partial risk premia into the economic literature.

Definition 2.1 *For u and v, the partial risk premia π_u and π_v for risk $\tilde{x}$ in the presence of risk $\tilde{y}$, are defined as*

$$Eu(w + \tilde{x}, \tilde{y}) = Eu(w + E\tilde{x} - \pi_u, \tilde{y}) \tag{1}$$

and

$$Ev(w + \tilde{x}, \tilde{y}) = Ev(w + E\tilde{x} - \pi_v, \tilde{y}). \tag{2}$$

The partial risk premia π_u and π_v are the maximal monetary amounts individuals u and v are willing to pay for removing one risk in the presence of a second risk. We derive necessary and sufficient conditions for comparative partial risk premia in the presence of a conditional expectation increasing background risk. Extension of the analysis to conditional expectation decreasing background risk is discussed later. Let us introduce two definitions of comparative risk aversion motivated by Ross (1981). The following definition uses $-\frac{u_{12}(w,y)}{u_1(w,y)}$ and $-\frac{v_{12}(w,y)}{v_1(w,y)}$ as local measures of correlation aversion.

Definition 2.2 *u is more cross Ross risk averse than v if and only if there exists $\lambda_1, \lambda_2 > 0$*

such that for all w, y and y'

$$\frac{u_{12}(w,y)}{v_{12}(w,y)} \geq \lambda_1 \geq \frac{u_1(w,y')}{v_1(w,y')} \tag{3}$$

and

$$\frac{u_{11}(w,y)}{v_{11}(w,y)} \geq \lambda_2 \geq \frac{u_1(w,y')}{v_1(w,y')}. \tag{4}$$

The interpretation of the sign of the second cross derivative goes back to De Finetti (1952) and has been studied and extended by Epstein and Tanny (1980); Richard (1975); Scarsini (1988) and Eeckhoudt et al. (2007). For example, Eeckhoudt *et al.* (2007) show that $u_{12} \leq 0$ is necessary and sufficient for an agent to be "correlation averse," meaning that a higher level of the second argument mitigates the detrimental effect of a reduction in the first argument. Agents are correlation averse if they always prefer a 50-50 gamble of a loss in x or a loss in y over another 50-50 gamble offering a loss in both x and y.

When $u(w,y) = U(w+y)$ in (3) and (4), we obtain the definition of comparative Ross risk aversion for mean independent risks. Here, we are interested in comparisons when the agents face two dependent risks, which is more general than mean independence. We consider the notion of conditional background risk. Two random variables are conditional risk dependent when the following definition is met:

Definition 2.3 *$\tilde{y}$ is a conditional expectation increasing (or decreasing) background risk for $\tilde{x}$ if $E[\tilde{x}|\tilde{y}=y]$ is increasing (or decreasing) in y.*

Tsetlin and Winkler (2005) define this measure of dependence as a positive (or negative) relation. They identify the conditions that satisfy the property of having $E[\tilde{x}|\tilde{y}=y]$ increasing (or decreasing) in y. Positive (or negative) relation is a stronger condition than positive (negative) correlation, but a weaker condition than affiliation (Milgrom and Weber, 1982). Moreover, the condition for positive relation is not symmetric to the condition for negative relation.

The following proposition provides an equivalent comparison between risk aversion and partial risk premium in the presence of conditional expectation increasing background risks.

Proposition 2.4 *For u, v with $u_1 > 0$, $v_1 > 0$, $v_{11} < 0$, $u_{11} < 0$, $u_{12} < 0$ and $v_{12} < 0$, the following three conditions are equivalent:*

(i) u is more cross Ross risk averse than v.

(ii) There exists $\phi : R \times R \to R$ with $\phi_1 \leq 0$, $\phi_{12} \leq 0$ and $\phi_{11} \leq 0$, and $\lambda > 0$ such that $u = \lambda v + \phi$.

(iii) $\pi_u \geq \pi_v$ *for* $\forall$ w *and* $(\tilde{x}, \tilde{y})$ *such that* $E[\tilde{x}|\tilde{y} = y]$ *is non-decreasing in* y.

Proof See the Appendix.

Proposition 2.4 shows that when an agent faces a conditional expectation increasing background risk, the cross Ross risk aversion relationship establishes an unambiguous equivalence between more risk aversion and a larger partial risk premium. We now present an example.

Example Suppose $u(x, y) = x + y - \beta e^{-(x+y)}$ and $v(x, y) = x + y - \beta e^{-(x+y)} - \frac{1}{2}x^2y^2$ where $\beta > 0$ and assume that $\tilde{x}$ and $\tilde{y}$ are scaled, so that $0 < x < 1$ and $0 < y < 1$. Hence $u(x, y) = \lambda v(x, y) + \phi(x, y)$ where $\lambda = 1$ and $\phi(x, y) = -\frac{1}{2}x^2y^2$. Because

$$u_1 = 1 + \beta e^{-(x+y)} > 0, \tag{5}$$

$$u_{11} = u_{12} = -\beta e^{-(x+y)} < 0, \tag{6}$$

$$v_1 = 1 + \beta e^{-(x+y)} - xy^2 > 0, \tag{7}$$

$$v_{11} = -\beta e^{-(x+y)} - y^2 < 0, \tag{8}$$

$$v_{12} = -\beta e^{-(x+y)} - 2xy < 0, \tag{9}$$

$$\phi_1 = -xy^2 < 0, \tag{10}$$

$$\phi_{11} = -y^2 < 0 \tag{11}$$

and

$$\phi_{12} = -2xy < 0, \tag{12}$$

from Proposition 2.4, we know that $u(x, y)$ is more cross Ross risk averse than $v(x, y)$.

3 Decreasing cross Ross risk aversion with respect to wealth

In this section, we examine how the partial risk premium for a given risk $\tilde{x}$ is affected by a change in initial wealth w, in the presence of a dependent background risk. Fully differentiating equation (1) with respect to w yields[1]

$$Eu_1(w + \tilde{x}, \tilde{y}) = Eu_1(w + E\tilde{x} - \pi_u, \tilde{y}) - \pi'(w)Eu_1(w + E\tilde{x} - \pi_u, \tilde{y}). \tag{13}$$

[1] Equation (13) has a univariate counterpart in Eeckhoudt and Kimball (1992).

Hence,

$$\pi'(w) = \frac{Eu_1(w + E\tilde{x} - \pi_u, \tilde{y}) - Eu_1(w + \tilde{x}, \tilde{y})}{Eu_1(w + E\tilde{x} - \pi_u, \tilde{y})}. \tag{14}$$

Thus, the partial risk premium is decreasing in wealth if and only if

$$Eh(w + E\tilde{x} - \pi_u, \tilde{y}) \geq Eh(w + \tilde{x}, \tilde{y}), \tag{15}$$

where $h \equiv -u_1$ is defined as minus the partial derivative of function u. Because $h_1 = -u_{11} \geq 0$, condition (15) simply states that the partial risk premium of agent h is larger than the partial risk premium of agent u. From Proposition 2.4, this is true if and only if h is more cross Ross risk averse than u. That is, $\exists \lambda_1, \lambda_2 > 0$, for all w,y and y', such that

$$\frac{h_{12}(w,y)}{u_{12}(w,y)} \geq \lambda_1 \geq \frac{h_1(w,y')}{u_1(w,y')} \tag{16}$$

and

$$\frac{h_{11}(w,y)}{u_{11}(w,y)} \geq \lambda_2 \geq \frac{h_1(w,y')}{u_1(w,y')}, \tag{17}$$

or, equivalently,

$$-\frac{u_{112}(w,y)}{u_{12}(w,y)} \geq \lambda_1 \geq -\frac{u_{11}(w,y')}{u_1(w,y')} \tag{18}$$

and

$$-\frac{u_{111}(w,y)}{u_{11}(w,y)} \geq \lambda_2 \geq -\frac{u_{11}(w,y')}{u_1(w,y')}. \tag{19}$$

Proposition 3.1 introduces $-\frac{u_{112}(w,y)}{u_{11}(w,y)}$ and $-\frac{u_{111}(w,y)}{u_{11}(w,y)}$ as local measurements of cross-prudence and prudence. These local measures of prudence are identical to the measure proposed by Kimball (1990). It is well known that, for the single-risk case, DARA is equivalent to the utility function $-u'(.)$ being more concave than $u(.)$ (see for example, Gollier, 2001). Proposition 3.1 is an extension of this result to bivariate risks under a conditional expectation increasing background risk.

We obtain the following proposition:

Proposition 3.1 *For u with $u_1 > 0$, $u_{11} < 0$, $u_{12} < 0$, $u_{111} \geq 0$ and $u_{112} \geq 0$, the following three conditions are equivalent:*

(i) the partial risk premium π_u, associated with any $(\tilde{x}, \tilde{y})$ such that $E[\tilde{x}|\tilde{y} = y]$ is non-decreasing in y, is decreasing in wealth;

(ii) There exists $\phi : R \times R \to R$ with $\phi_1 \leq 0$, $\phi_{12} \leq 0$ and $\phi_{11} \leq 0$, and $\lambda > 0$ such that $-u_1 = \lambda u + \phi$;

(iii) $\exists \lambda_1, \lambda_2 > 0$, *for all* w, y *and* y', *such that*

$$-\frac{u_{112}(w,y)}{u_{12}(w,y)} \geq \lambda_1 \geq -\frac{u_{11}(w,y')}{u_1(w,y')} \tag{20}$$

and

$$-\frac{u_{111}(w,y)}{u_{11}(w,y)} \geq \lambda_2 \geq -\frac{u_{11}(w,y')}{u_1(w,y')}. \tag{21}$$

The proof of Proposition 3.1 is obtained by using (13) to (19).

An interpretation of the sign of u_{112} is provided by Eeckhoudt et al. (2007), who show that $u_{112} > 0$ is a necessary and sufficient condition for "cross-prudence in its second argument," meaning that a higher level of the second argument y (or y') mitigates the detrimental effect of the monetary risk.

Example Suppose $u(x,y) = x + y - \beta e^{-(x+y)}$ where $0 < \beta < 1$. So

$$u_1 = 1 + \beta e^{-(x+y)} > 0. \tag{22}$$

Define $\phi(x,y) = -u_1(x,y) - \frac{1}{2}u(x,y) = -1 - \beta e^{-(x+y)} - \frac{1}{2}(x + y - \beta e^{-(x+y)})$. Given that

$$u_{11} = u_{12} = -\beta e^{-(x+y)} < 0, \tag{23}$$

$$u_{111} = u_{112} = \beta e^{-(x+y)} > 0, \tag{24}$$

$$\phi_1 = \beta e^{-(x+y)} - \frac{1}{2} - \frac{1}{2}\beta e^{-(x+y)} = -\frac{1}{2} + \frac{1}{2}\beta e^{-(x+y)} < 0 \tag{25}$$

and

$$\phi_{11} = \phi_{12} = -\frac{1}{2}\beta e^{-(x+y)} < 0, \tag{26}$$

from Proposition 3.1, we know that $u(x,y)$ exhibits decreasing cross Ross risk aversion with respect to wealth.

4 Conditional expectation decreasing background risk

There are economic applications where negative dependence is more convenient. If $E[\tilde{x}|\tilde{y} = y]$ is non-increasing in y, then $E[-\tilde{x}|\tilde{y} = y]$ is non-decreasing in y. We can define $\bar{u}(x,y) = u(-x,y)$ and $\bar{v}(x,y) = v(-x,y)$. Then Propositions 2.4 and 3.1 can be extended to $\bar{u}(x,y)$ and $\bar{v}(x,y)$ directly. More specifically, we can propose the following results.

Proposition 4.1 *For $\bar{u}$, $\bar{v}$ with $\bar{u}_1 > 0$, $\bar{v}_1 > 0$, $\bar{u}_{11} < 0$, $\bar{v}_{11} < 0$, $\bar{u}_{12} < 0$ and $\bar{v}_{12} < 0$, the following three conditions are equivalent:*

(i) $\bar{u}$ is more cross Ross risk averse than $\bar{v}$.

(ii) There exists $\phi : R \times R \to R$ with $\phi_1 \leq 0$, $\phi_{12} \leq 0$ and $\phi_{11} \leq 0$, and $\lambda > 0$ such that $\bar{u} = \lambda \bar{v} + \phi$.

(iii) $\pi_{\bar{u}} \geq \pi_{\bar{v}}$ for $\forall$ w and $(\tilde{x}, \tilde{y})$ such that $E[\tilde{x}|\tilde{y} = y]$ is non-increasing in y.

and

Proposition 4.2 *For $\bar{u}$ with $\bar{u}_1 > 0$, $\bar{u}_{11} < 0$, $\bar{u}_{12} < 0$, $\bar{u}_{111} \geq 0$ and $\bar{u}_{112} \geq 0$, the following three conditions are equivalent:*

(i) the partial risk premium $\pi_{\bar{u}}$ associated with any $(\tilde{x}, \tilde{y})$ such that $E[\tilde{x}|\tilde{y} = y]$ is non-increasing in y, is decreasing in wealth;

(ii) There exists $\phi : R \times R \to R$ with $\phi_1 \leq 0$, $\phi_{12} \leq 0$ and $\phi_{11} \leq 0$, and $\lambda > 0$ such that $-\bar{u}_1 = \lambda \bar{u} + \phi$;

(iii) $\exists \lambda_1, \lambda_2 > 0$, for all w, y and y', such that

$$-\frac{\bar{u}_{112}(w, y)}{\bar{u}_{12}(w, y)} \geq \lambda_1 \geq -\frac{\bar{u}_{11}(w, y')}{\bar{u}_1(w, y')} \tag{27}$$

and

$$-\frac{\bar{u}_{111}(w, y)}{\bar{u}_{11}(w, y)} \geq \lambda_2 \geq -\frac{\bar{u}_{11}(w, y')}{\bar{u}_1(w, y')}. \tag{28}$$

5 Comparative risk aversion in the presence of a financial background risk

In the economic literature, there has been much attention on the financial background risk. For additive financial background risk, we refer to Doherty and Schlesinger (1983a,b, 1986), Kischka (1988), Eeckhoudt and Kimball (1992), Eeckhoudt and Gollier, (2000), Schlesinger (2000), Gollier (2001), Eeckhoudt et al. (2007) and Franke et al. (2011). For multiplicative financial background risk, we refer to Franke et al. (2006, 2011). In this section, we consider some examples to illustrate the use of Propositions 2.4 and 3.1 in the framework of additive or multiplicative background risks.

5.1 Additive background risk

First, we show that Proposition 2.4 allows us to extend the results of Ross (1981) for an additive background risk. Note that, for an additive background risk $\tilde{y}$, we have

$$u(w, y) = U(w + y) \tag{29}$$

and

$$v(w, y) = V(w + y). \tag{30}$$

Here w can be the wealth of an agent and y an increment to wealth, i.e., income.

Since,

$$u_1 = U' \quad , \quad u_{11} = u_{12} = U'' \ \ and \ \ u_{111} = u_{112} = U''' \tag{31}$$

and

$$v_1 = V' \quad , \quad v_{11} = v_{12} = V'' \ \ and \ \ v_{111} = v_{112} = V'''. \tag{32}$$

conditions (3) and (4) imply

$$\frac{U''(w+y)}{V''(w+y)} \geq \lambda \geq \frac{U'(w+y')}{V'(w+y')} \quad for \ \ all \ w \ , y \ and \ y'. \tag{33}$$

Then, Proposition 2.4, (31), (32) and (33) immediately entail the following result.

Corollary 5.1 *For $u(w, y) = U(w + y)$, $v(w, y) = V(w + y)$ with $U' > 0$, $V' > 0$, $U'' < 0$ and $V'' < 0$, the following two conditions are equivalent:*

(i) $\exists \lambda > 0$

$$\frac{U''(w+y)}{V''(w+y)} \geq \lambda \geq \frac{U'(w+y')}{V'(w+y')} \quad for \ \ all \ w \ , y \ and \ y'. \tag{34}$$

(ii) $\pi_u \geq \pi_v$ for $\forall$ w and $(\tilde{x}, \tilde{y})$ such that $E[\tilde{x}|\tilde{y} = y]$ is non-decreasing in y.

Conditions (20) and (21) imply, for all w, y and y',

$$-\frac{U'''(w+y)}{U''(w+y)} \geq \lambda \geq -\frac{U''(w+y')}{U'(w+y')} \tag{35}$$

From Proposition 3.1, (31), (32) and (33), we obtain the following corollary:

Corollary 5.2 *For $u(w, y) = U(w + y)$, with $U' > 0$, $U'' < 0$ and $U''' > 0$, the following two conditions are equivalent:*

(i) the partial risk premium associated to any $(\tilde{x}, \tilde{y})$ such that $E[\tilde{x}|\tilde{y} = y]$ is non-decreasing in y, is decreasing in wealth.

(ii) $\exists\lambda > 0$, *for all* w, y *and* y',

$$-\frac{U'''(w+y)}{U''(w+y)} \geq \lambda \geq -\frac{U''(w+y')}{U'(w+y')} \tag{36}$$

In Corollary 5.2, the condition for decreasing risk premia under conditional expectation increasing risks is equivalent to that for a first-order stochastic dominance (FSD) improvement in an independent background risk to decrease the risk premium, as shown by Eeckhoudt et al. (1996).

Ross (1981) proposed the following results

Proposition 5.3 *(Ross (1981, Theorem 3)) For* $u(w, y) = U(w + y)$, $v(w, y) = V(w + y)$ *with* $U' > 0$, $V' > 0$, $U'' < 0$ *and* $V'' < 0$, *the following two conditions are equivalent:*

(i) $\exists\lambda > 0$

$$\frac{U''(w+y)}{V''(w+y)} \geq \lambda \geq \frac{U'(w+y')}{V'(w+y')} \quad for \ \ all \ w \ ,y \ and \ y'. \tag{37}$$

(ii) $\pi_u \geq \pi_v$ *for* $\forall$ w, *any zero-mean risk* $\tilde{x}$ *and* $\tilde{y}$ *with* $E[\tilde{x}|y] = E\tilde{x} = 0$.

Proposition 5.4 *(Ross (1981, Theorem 4)) For* $u(w, y) = U(w + y)$, *with* $U' > 0$, $U'' < 0$ *and* $U''' > 0$, *the partial risk premium associated to any zero-mean risk* $\tilde{x}$ *with* $E[\tilde{x}|y] = 0$ *is decreasing in wealth if and only if,* $\exists\lambda > 0$, *for all* w, y *and* y',

$$-\frac{U'''(w+y)}{U''(w+y)} \geq \lambda \geq -\frac{U''(w+y')}{U'(w+y')} \tag{38}$$

It is easy to see that Corollaries 5.1 and 5.2 generalize Ross' conclusions to a more general setting.

5.2 Multiplicative background risk

For a multiplicative background risk $\tilde{y}$, we have

$$u(w, y) = U(wy) \tag{39}$$

and

$$v(w, y) = V(wy). \tag{40}$$

Here w may represent the quantity invested in a risky asset and y may represent the random return per unit of wealth invested in the asset.

Since,

$$u_1 = yU', \quad u_{11} = y^2U'', \quad u_{12} = U' + wyU'', \quad u_{111} = y^3U''' \quad and \quad u_{112} = 2yU''^2U''' \tag{41}$$

and

$$v_1 = yV', \quad v_{11} = y^2V'', \quad v_{12} = V' + wyV'', \quad v_{111} = y^3V''' \quad and \quad v_{112} = 2yV''^2V''', \tag{42}$$

conditions (3) and (4) imply, $\exists \lambda_1, \lambda_2 > 0$, for all w,y and y',

$$\frac{U'(wy) + wyU''(wy)}{V'(wy) + wyV''(wy)} \geq \lambda_1 \geq \frac{U'(wy')}{V'(wy')} \tag{43}$$

and

$$\frac{U''(wy)}{V''(wy)} \geq \lambda_2 \geq \frac{U'(wy')}{V'(wy')}. \tag{44}$$

Then, from Proposition 2.4, (41), (42), (43) and (44), we obtain

Corollary 5.5 *For $u(w,y) = U(wy)$, $v(w,y) = V(wy)$ with $U' > 0$, $V' > 0$, $U'' < 0$ and $V'' < 0$, the following two conditions are equivalent:*

(i) $\exists \lambda_1, \lambda_2 > 0$, for all w,y and y',

$$\frac{U'(wy) + wyU''(wy)}{V'(wy) + wyV''(wy)} \geq \lambda_1 \geq \frac{U'(wy')}{V'(wy')} \tag{45}$$

and

$$\frac{U''(wy)}{V''(wy)} \geq \lambda_2 \geq \frac{U'(wy')}{V'(wy')}. \tag{46}$$

(ii) $\pi_u \geq \pi_v$ for $\forall$ w and $(\tilde{x}, \tilde{y})$ such that $E[\tilde{x}|\tilde{y} = y]$ is non-decreasing in y.

Since

$$\begin{aligned} & \frac{U'(wy) + wyU''(wy)}{V'(wy) + wyV''(wy)} \\ = & \frac{U''(wy)(\frac{U'(wy)}{U''(wy)} + wy)}{V''(wy)(\frac{V'(wy)}{V''(wy)} + wy)} \\ = & \frac{U''(wy)(wy - \frac{1}{RA_U(wy)})}{V''(wy)(wy - \frac{1}{RA_V(wy)})}, \end{aligned} \tag{47}$$

where $RA_U(wy) = -\frac{U''(wy)}{U'(wy)}$ and $RA_V(wy) = -\frac{V''(wy)}{V'(wy)}$ are indexes of absolute risk aversion. We can obtain a more short cut sufficient condition from Corollary 5.5.

Corollary 5.6 *For $u(w,y) = U(wy)$, $v(w,y) = V(wy)$ with $w > 0$, $\tilde{y} > 0$ almost surely, $U' > 0$, $V' > 0$, $U'' < 0$ and $V'' < 0$, If $\exists\lambda > 0$, for all w,y and y',*

$$\frac{U''(wy)}{V''(wy)} \geq \lambda \geq \frac{U'(wy')}{V'(wy')}, \tag{48}$$

then $\pi_u \geq \pi_v$ for $\forall$ w and $(\tilde{x}, \tilde{y})$ such that $E[\tilde{x}|\tilde{y} = y]$ is non-decreasing in y.

Proof From the above argument, we know that for all w,y and y',

$$\frac{U''(wy)}{V''(wy)} \geq \lambda \geq \frac{U'(wy')}{V'(wy')} \tag{49}$$

and $RA_U(wy) \geq RA_V(wy)$ implies that $\pi_u \geq \pi_v$ for $\forall$ w and $(\tilde{x}, \tilde{y})$ such that $E[\tilde{x}|\tilde{y} = y]$ is non-decreasing in y. Using the fact that "U is more Ross risk averse than $V \Rightarrow RA_U(wy) \geq RA_V(wy)$", we obtain the result. Q.E.D.

Corollary 5.6 states that "more Ross risk aversion" is a sufficient condition to order partial risk premium in the presence of an conditional expectation increasing multiplicative background risk.

From Proposition 3.1, we obtain

Corollary 5.7 *For $u(w,y) = U(wy)$, with $U' > 0$, $U'' < 0$ and $U''' > 0$,The partial risk premiums associated to any $(\tilde{x}, \tilde{y})$ such that $E[\tilde{x}|\tilde{y} = y]$ is non-decreasing in y, is decreasing in wealth if and only if, $\exists\lambda_1, \lambda_2 > 0$, for all w,y and y',*

$$-\frac{2yU''^2U'''(wy)}{U'(wy) + wyU''(wy)} \geq \lambda_1 \geq -\frac{y'U''(wy')}{U'(wy')} \tag{50}$$

and

$$-\frac{yU'''(wy)}{U''(wy)} \geq \lambda_2 \geq -\frac{y'U''(wy')}{U'(wy')}. \tag{51}$$

Since

$$\begin{aligned} & -\frac{2yU''^2U'''(wy)}{U'(wy) + wyU''(wy)} \\ = & -\frac{yU'''(wy)(2\frac{U''(wy)}{U'''(wy)} + wy)}{U''(wy)(\frac{U'(wy)}{U''(wy)} + wy)} \\ = & -\frac{yU'''(wy)(wy - 2\frac{1}{P_U(wy)})}{U''(wy)(wy - \frac{1}{RA_U(wy)})}, \end{aligned} \tag{52}$$

where $P_U(wy) = -\frac{U'''(wy)}{U''(wy)}$ is the index of absolute prudence. We can obtain a more short cut sufficient condition from Corollary 5.7.

Corollary 5.8 *For $u(w,y) = U(wy)$, with $w > 0$, $\tilde{y} > 0$ almost surely, $U' > 0$, $U'' < 0$ and $U''' > 0$, The partial risk premium associated to any risk $(\tilde{x}, \tilde{y})$ such that $E[\tilde{x}|\tilde{y} = y]$ is non-decreasing in y, is decreasing in wealth if , $\exists\lambda > 0$, for all w, y and y',*

$$-\frac{yU'''(wy)}{U''(wy)} \geq \lambda \geq -\frac{y'U''(wy')}{U'(wy')} \tag{53}$$

and $P_U(wy) \geq 2RA_U(wy)$.

Moreover, (53) can be multiplied by w on both sides to obtain the results in terms of measures of relative risk aversion and relative prudence:

$$-\frac{wyU'''(wy)}{U''(wy)} \geq \lambda \geq -\frac{wy'U''(wy')}{U'(wy')}, \tag{54}$$

which implies "min relative prudence $\geq$ max relative risk aversion". While, in the literature, $P_U \geq 2RA_U$ is an important condition for risk vulnerability (see Gollier 2001, p129), Corollary 5.8 shows that $P_U \geq 2RA_U$ is also an important condition for comparative risk aversion in the presence of a conditional expectation increasing multiplicative background risk.

6 Application: comparing the welfare cost of business cycles

We consider the following question: What is the effect on welfare of eliminating all consumption variability? Consider a single consumer, endowed with initial wealth w and the stochastic consumption stream $\tilde{c}$ with $E\tilde{c} > 0$. Preferences over such consumption are assumed to be

$$Eu(w + \tilde{c}, \tilde{y}), \tag{55}$$

where $\tilde{y}$ is a background risk such as the returns of the stock index funds owned by the consumer.

A risk-averse consumer would prefer a deterministic consumption to a risky one with the same mean, under certain conditions. We quantify this utility difference by multiplying the risky consumption by the constant factor $1 + \lambda_u$. We choose λ_u so that the household is indifferent between the deterministic consumption and the compensated risky one. That is, λ_u^* solves

$$Eu(w + (1 + \lambda_u^*)\tilde{c}, \tilde{y}) = Eu(w + E\tilde{c}, \tilde{y}). \tag{56}$$

Lucas (1987; 2003) defines this compensation parameter λ_u^* as the welfare gain (or welfare loss) from eliminating consumption risk. Because for $\forall\lambda_u$, we have

$$\begin{aligned} Eu(w + (1 + \lambda_u)\tilde{c}, \tilde{y}) &= Eu(w + (1 + \lambda_u)(E\tilde{c} - \pi_u), \tilde{y}) \\ &= Eu(w + E\tilde{c} + \lambda_u E\tilde{c} - (1 + \lambda_u)\pi_u, \tilde{y}), \end{aligned}$$

hence $\lambda_u^* = \frac{\pi_u}{E\tilde{c}-\pi_u}$ and λ_u^* is increasing in π_u. Let λ_v^* solve the following equation:

$$Ev(w + (1 + \lambda_v^*)\tilde{c}, \tilde{y}) = Ev(w + E\tilde{c}, \tilde{y}). \tag{57}$$

From propositions 2.4 and 3.1, we obtain the following result.

Proposition 6.1 *Suppose $E[\tilde{c}|\tilde{y} = y]$ is non-decreasing in y. For u, v with $u_1 > 0$, $v_1 > 0$, $v_{11} < 0$, $u_{11} < 0$, $u_{12} < 0$, $v_{12} < 0$, $u_{111} \geq 0$ and $u_{112} \geq 0$,*

(i) If u is more cross Ross risk averse than v, then $\lambda_u^* \geq \lambda_v^*$.

(ii) If u exhibits decreasing cross Ross risk aversion with respect to wealth, then λ_u^* is decreasing in w.

Example Suppose $u(x, y) = x + y - \beta e^{-(x+y)}$ and $v(x, y) = x + y - \beta e^{-(x+y)} - \frac{1}{2}x^2y^2$ where $\beta > 0$ and assume that $\tilde{x}$ and $\tilde{y}$ are scaled, so that $0 < x < 1$ and $0 < y < 1$. Then $\lambda_u^* \geq \lambda_v^*$ when $E[\tilde{c}|\tilde{y} = y]$ is non-decreasing in y.

Example Suppose $u(x, y) = x + y - \beta e^{-(x+y)}$ where $0 < \beta < 1$. Then λ_u^* is decreasing in wealth when $E[\tilde{c}|\tilde{y} = y]$ is non-decreasing in y.

While there is a great tradition of quantitative public finance that applies (56) and (57) to compute the welfare loss of business cycles risk, Proposition 6.1 provides a comparative welfare analysis of public policies in the presence of a background risk.

7 Decreasing cross Ross risk aversion and n-switch independence property

Bell (1988) argues that agents are likely to be characterized by a utility function satisfying the one-switch rule: there exists at most one critical wealth level at which the decision-maker switches from preferring one alternative to the other. He shows that the linex function (linear plus exponential) is the only relevant utility function family if one adds to the one-switch rule some very reasonable requirements. This utility function has been studied by Bell and Fishburn (2001), Sandvik and Thorlund-Petersen (2010), Abbas and Bell (2011) and Tsetlin and Winkler (2009, 2012). In a recent paper, Denuit *et al.* (2011b) show that Ross' stronger measure of risk aversion gives rise to the linex univariate utility function. They thus provide not only a utility function family but also some intuitive and convenient properties for Ross' measure.

Abbas and Bell (2011) extend the one-switch independence property to two-attribute utility functions, and propose a new independence assumption based on the one-switch property: n-switch independence (see Tsetlin and Winkler, 2012, for a similar extension).

Definition (Abbas and Bell 2011) For utility function $u(x, y)$, X is n-switch independent of Y if two gambles $\tilde{x}_1$ and $\tilde{x}_2$ can switch in preference at most n times as Y progresses from its lowest to its highest value.

They provide the following two propositions:

Proposition 7.1 *(Abbas and Bell 2011) X is one-switch independent of Y if and only if*

$$u(x, y) = g_0(y) + f_1(x)g_1(y) + f_2(x)g_2(y), \tag{58}$$

where $g_1(y)$ has a constant sign, and $g_2(y) = g_1(y)\phi(y)$ for some monotonic function ϕ.

Proposition 7.2 *(Abbas and Bell 2011) If X is n-switch independent of Y, then there exist some functions f_i, g_i such that*

$$u(x, y) = g_0(y) + \sum_{i=1}^{n+1} f_i(x)g_i(y). \tag{59}$$

We now show that the one-switch property of Proposition 7.1 is directly connected to Proposition 3.1. We also demonstrate that (59) is a utility function that satisfies the decreasing cross Ross risk aversion condition proposed in Section 3.

¿From Proposition 3.1 we know that the partial risk premium π_u, associated with any non-decreasing conditional expectation, is decreasing in wealth, if and only if there exists $\phi : R \times R \rightarrow R$ with $\phi_1 \leq 0$, $\phi_{12} \leq 0$ and $\phi_{11} \leq 0$, and $\lambda > 0$ such that

$$-u_1(x, y) = \lambda u(x, y) + \phi(x, y). \tag{60}$$

Solving the above differential equation implies that u is of the form

$$u(x, y) = -\int_{-\infty}^{x} e^{\lambda t}\phi(t, y)dt e^{-\lambda x}. \tag{61}$$

If we take $\phi(x, y) = -H(x)J(y)$ such that $J(y)$ has a constant sign, then we get

$$\begin{aligned} u(x, y) &= \int_{-\infty}^{x} e^{\lambda t}H(t)dte^{-\lambda x}J(y) \\ &= [\frac{1}{\lambda}e^{\lambda x}H(x) - \frac{1}{\lambda}\int_{-\infty}^{x} e^{\lambda t}H'(t)dt]e^{-\lambda x}J(y) \\ &= \frac{1}{\lambda}H(x)J(y) - \frac{1}{\lambda}\int_{-\infty}^{x} e^{\lambda t}H'(t)dte^{-\lambda x}J(y). \end{aligned} \tag{62}$$

Defining $g_1(y) = g_2(y) = \frac{1}{\lambda}J(y)$, $f_1(x) = H(x)$ and $f_2(x) = -\int_{-\infty}^{x} e^{\lambda t}H'(t)dte^{-\lambda x}$, we recognize the functional form in Proposition 7.1.

Integrating the integral term of (62) by parts again and again, we obtain

$$\begin{aligned} u(x,y) &= [\sum_{i=1}^{n} e^{\lambda x}\frac{(-1)^{i-1}H^{(i-1)}(x)}{\lambda^i} + \frac{1}{\lambda^n}\int_{-\infty}^{x} e^{\lambda t}(-1)^n H^{(n)}(t)dt]e^{-\lambda x}J(y) \qquad (63) \\ &= \sum_{i=1}^{n} J(y)\frac{(-1)^{i-1}H^{(i-1)}(x)}{\lambda^i} + \frac{1}{\lambda^n}\int_{-\infty}^{x} e^{\lambda t}(-1)^n H^{(n)}(t)dte^{-\lambda x}J(y) \\ &= \sum_{i=1}^{n+1} f_i(x)g_i(y), \end{aligned}$$

where $f_i(x) = (-1)^{(i-1)}H^{(i-1)}(x)$ for $i = 1, ..., n$, $f_{n+1}(x) = \int_{-\infty}^{x} e^{\lambda t}(-1)^n H^{(n)}(t)dte^{-\lambda x}$, $g_i(y) = \frac{1}{\lambda^i}J(y)$ for $i = 1, .., n$ and $g_{n+1}(y) = \frac{1}{\lambda^n}J(y)$. Therefore we obtain the functional form in Proposition 7.2 from decreasing cross Ross risk aversion. (63) shows that with decreasing cross Ross risk aversion we may get the same function forms suggested by n-switch independence. Our result thus provides a connection between decreasing cross Ross risk aversion and n-switch independence.

Finally, we point out that utilities in Proposition 3.1 might be consistent with n-switch independence, but they might not be. In this section, we only discuss under which conditions a n-switch utility would correspond to Proposition 3.1.

8 Conclusion

In this paper we consider expected-utility preferences in a bivariate setting. The analysis focuses on random variables that satisfy the conditional background risk. The main focus is on the risk premium for removing one of the risks in the presence of a second dependent risk. To this end, we extend Ross' (1981) contribution by defining the concept of "cross Ross risk aversion." We derive several equivalence theorems relating measures of risk premia with measures of risk aversion. We then apply our results to comparing the welfare cost of business cycles and we provide evidence of a direct relationship between decreasing cross Ross risk aversion and the n-switch independence property. The analysis and the index of risk aversion in this paper may be instrumental in obtaining comparative static predictions in various economic applications in the presence of a background risk.

Other potential applications are the equity premium puzzle and the financial market participation puzzle. These two puzzles have been analyzed in presence of an independent background risk (Weil, 1992; Gollier and Schlesinger, 2002; Heaton and Lucas, 2000). These risks can be dependent for many decision makers such as an IBM employee who is considering buying IBM stock or a non-permanent employee who owns units of stock index funds and whose job is subject to business cycles. Extending the above models to dependent background risk is important because the introduction of an independent background risk has not yet solve the two puzzles.

9 Appendix: Proof of Proposition 2.4

Proof (i) implies (ii): We note that

$$\frac{u_{12}(w,y)}{v_{12}(w,y)} \geq \lambda_1 \geq \frac{u_1(w,y')}{v_1(w,y')} \Leftrightarrow \frac{-u_{12}(w,y)}{-v_{12}(w,y)} \geq \lambda_1 \geq \frac{u_1(w,y')}{v_1(w,y')}. \tag{64}$$

$$\frac{u_{11}(w,y)}{v_{11}(w,y)} \geq \lambda_2 \geq \frac{u_1(w,y')}{v_1(w,y')} \Leftrightarrow \frac{-u_{11}(w,y)}{-v_{11}(w,y)} \geq \lambda_2 \geq \frac{u_1(w,y')}{v_1(w,y')}. \tag{65}$$

Defining $\phi = u - \lambda v$, where $\lambda = \min\{\lambda_1, \lambda_2\}$, and differentiating, one obtains $\phi_1 = u_1 - \lambda v_1$, $\phi_{12} = u_{12} - \lambda v_{12}$ and $\phi_{11} = u_{11} - \lambda v_{11}$, then (64) and (65) imply that $\phi_1 \leq 0$, $\phi_{12} \leq 0$ and $\phi_{11} \leq 0$.

(ii) implies (iii): From Theorem 2 of Finkelshtain et al. (1999), we know that,

(a) $\phi_{11} \leq 0$ and $\phi_{12} \leq 0 \Leftrightarrow E\phi(w + \tilde{x}, \tilde{y}) \leq E\phi(w + E\tilde{x}, \tilde{y})$ for any risk $(\tilde{x}, \tilde{y})$ such that $E[\tilde{x}|\tilde{y} = y]$ is non-decreasing in y;

(b) when $v_1 \geq 0$, $v_{11} \leq 0$ and $v_{12} \leq 0$ if and only if $\pi_v \geq 0$ for any risk $(\tilde{x}, \tilde{y})$ such that $E[\tilde{x}|\tilde{y} = y]$ is non-decreasing in y.

Because $\pi_v \geq 0$, we have $\phi_1 \leq 0 \Rightarrow \phi(w, y) \leq \phi(w - \pi_v, y)$.

The following proof is as in Ross (1981):

$$\begin{aligned} Eu(w + E\tilde{x} - \pi_u, \tilde{y}) &= Eu(w + \tilde{x}, \tilde{y}) \\ &= E[\lambda v(w + \tilde{x}, \tilde{y}) + \phi(w + \tilde{x}, \tilde{y})] \\ &= \lambda Ev(w + E\tilde{x} - \pi_v, \tilde{y}) + E\phi(w + \tilde{x}, \tilde{y}) \\ &\leq \lambda Ev(w + E\tilde{x} - \pi_v, \tilde{y}) + E\phi(w + E\tilde{x}, \tilde{y}) \\ &\leq \lambda Ev(w + E\tilde{x} - \pi_v, \tilde{y}) + E\phi(w + E\tilde{x} - \pi_v, \tilde{y}) \\ &= Eu(w + E\tilde{x} - \pi_v, \tilde{y}). \end{aligned} \tag{66}$$

Because $u_1 > 0$, $\pi_u \geq \pi_v$.

(iii) implies (i): We prove this claim by contradiction. Suppose that there exists some w, y and y' such that $\frac{u_{12}(w,y)}{v_{12}(w,y)} < \frac{u_1(w,y')}{v_1(w,y')}$. Because u_1, v_1, u_{12} and v_{12} are continuous, we have

$$\frac{u_{12}(w,y)}{v_{12}(w,y)} < \frac{u_1(w,y')}{v_1(w,y')} \quad for \quad (w,y),(w,y') \in [m_1,m_2]\times[n_1,n_2], \tag{67}$$

which implies

$$\frac{-u_{12}(w,y)}{-v_{12}(w,y)} < \frac{u_1(w,y')}{v_1(w,y')} \quad for \quad (w,y),(w,y') \in [m_1,m_2]\times[n_1,n_2], \tag{68}$$

and then

$$\frac{v_1(w,y')}{-v_{12}(w,y)} < \frac{u_1(w,y')}{-u_{12}(w,y)} \quad for \quad (w,y),(w,y') \in [m_1,m_2]\times[n_1,n_2]. \tag{69}$$

If $G_Y(y)$ is the marginal distribution function of $\tilde{y}$. such that $G_Y(y)$ has positive support on interval $[n_1,n_2]$ then we have

$$\frac{Ev_1(w,\tilde{y})}{-v_{12}(w,y)} < \frac{Eu_1(w,\tilde{y})}{-u_{12}(w,y)} \quad for \quad (w,y) \in [m_1,m_2]\times[n_1,n_2], \tag{70}$$

which can be written as

$$\frac{u_{12}(w,y)}{Eu_1(w,\tilde{y})} > \frac{v_{12}(w,y)}{Ev_1(w,\tilde{y})} \quad for \quad (w,y) \in [m_1,m_2]\times[n_1,n_2]. \tag{71}$$

Let us consider $w_0 \in [m_1,m_2]$ and $\tilde{x} = k\tilde{z}$ with $k > 0$, where $\tilde{z}$ is a zero-mean risk with a distribution function $G(z,y)$ such that $G_Z(\tilde{z} \le z|\tilde{y} = y)$ is non-increasing in y. We notice that

(a) $G_Z(\tilde{z} \le z|\tilde{y} = y)$ is non-increasing in $y \Rightarrow E[\tilde{z}|\tilde{y} = y]$, is non-decreasing in y;

(b) $G_Z(\tilde{z} \le z|\tilde{y} = y)$ is non-increasing in $y \Rightarrow G(\tilde{y} \le y, \tilde{z} \le z) \ge G_Y(\tilde{y} \le y)G_Z(\tilde{z} \le z)$ (see Lehmann 1966, Lemma 4).

Let $\pi_u(k)$ denote its associated partial risk premium, which is

$$Eu(w_0 + k\tilde{z}, \tilde{y}) = Eu(w_0 - \pi_u(k), \tilde{y}). \tag{72}$$

Differentiating the above equality with respect to k yields

$$E\tilde{z}u_1(w_0 + k\tilde{z}, \tilde{y}) = -\pi_u'(k)Eu_1(w_0 - \pi_u(k), \tilde{y}). \tag{73}$$

Observing that $\pi_u(0) = 0$, we get

$$\begin{aligned} \pi_u'(0) &= -\frac{E\tilde{z}u_1(w_0,\tilde{y})}{Eu_1(w_0,\tilde{y})} \qquad (74) \\ &= -\frac{E\tilde{z}Eu_1(w_0,\tilde{y}) + Cov(\tilde{z}, u_1(w_0,\tilde{y}))}{Eu_1(w_0,\tilde{y})} \\ &= -\frac{Cov(\tilde{z}, u_1(w_0,\tilde{y}))}{Eu_1(w_0,\tilde{y})} \\ &= -\frac{\int\int[G(z,y) - G_Z(z)G_Y(y)]dzd_yu_1(w_0,y)}{Eu_1(w_0,\tilde{y})} \quad (by\ Cuadras\ 2002,\ Theorem\ 1) \\ &= -\int\int[G(z,y) - G_Z(z)G_Y(y)]\frac{u_{12}(w_0,y)}{Eu_1(w_0,\tilde{y})}dzdy \end{aligned}$$

Similarly, for v we have

$$\pi_v'(0) = -\int\int [G(z,y) - G_Z(z)G_Y(y)] \frac{v_{12}(w_0, y)}{Ev_1(w_0, \tilde{y})} dzdy. \tag{75}$$

Now π_u and π_v can be written as the forms of Taylor expansion around $k = 0$:

$$\pi_u(k) = -k\int\int [G(z,y) - G_Z(z)G_Y(y)] \frac{u_{12}(w_0, y)}{Eu_1(w_0, \tilde{y})} dzdy + o(k) \tag{76}$$

and

$$\pi_v(k) = -k\int\int [G(z,y) - G_Z(z)G_Y(y)] \frac{v_{12}(w_0, y)}{Ev_1(w_0, \tilde{y})} dzdy + o(k). \tag{77}$$

Then, from (71), we know that, when $k \to 0$, we get $\pi_u < \pi_v$ for $G(z,y)$ such that $G_Y(y)$ has positive support on interval $[n_1, n_2]$ and $G(z,y) - G_Z(z)G_Y(y)$ is positive on domain $[m_1, m_2] \times [n_1, n_2]$. This is a contradiction.

Now let us turn to the other condition. Suppose that there exists some w, y and y' such that $\frac{u_{11}(w,y)}{v_{11}(w,y)} < \frac{u_1(w,y')}{v_1(w,y')}$. Because u_1, v_1, u_{11} and v_{11} are continuous, we have

$$\frac{u_{11}(w,y)}{v_{11}(w,y)} < \frac{u_1(w,y')}{v_1(w,y')} \quad for \quad (w,y),(w,y') \in [m_1', m_2'] \times [n_1', n_2'], \tag{78}$$

which implies

$$\frac{-u_{11}(w,y)}{-v_{11}(w,y)} < \frac{u_1(w,y')}{v_1(w,y')} \quad for \quad (w,y),(w,y') \in [m_1', m_2'] \times [n_1', n_2'], \tag{79}$$

and then

$$\frac{-u_{11}(w,y)}{u_1(w,y')} < \frac{-v_{11}(w,y)}{v_1(w,y')} \quad for \quad (w,y),(w,y') \in [m_1', m_2'] \times [n_1', n_2']. \tag{80}$$

If $G(x,y)$ is a distribution function and $G_Y(y)$ has positive support on interval $[n_1', n_2']$, then we have

$$\frac{-Eu_{11}(w,\tilde{y})}{u_1(w,y')} < \frac{-Ev_{11}(w,\tilde{y})}{v_1(w,y')} \quad for \quad (w,y') \in [m_1', m_2'] \times [n_1', n_2'] \tag{81}$$

and

$$\frac{-Eu_{11}(w,\tilde{y})}{Eu_1(w,\tilde{y})} < \frac{-Ev_{11}(w,\tilde{y})}{Ev_1(w,\tilde{y})}. \tag{82}$$

Let us consider $w_0 \in [m_1', m_2']$ and $\tilde{x} = k\tilde{z}$, where $\tilde{z}$ is a zero-mean risk and $\tilde{z}$ and $\tilde{y}$ are independent. Let $\pi_u(k)$ denote its associated partial risk premium, which is

$$Eu(w_0 + k\tilde{z}, \tilde{y}) = Eu(w_0 - \pi_u(k), \tilde{y}). \tag{83}$$

Differentiating the equality above with respect to k yields

$$E\tilde{z}u_1(w_0 + k\tilde{z}, \tilde{y}) = -\pi_u'(k)Eu_1(w_0 - \pi_u(k), \tilde{y}), \tag{84}$$

and so $\pi'_u(0) = 0$ because $E\tilde{z} = 0$. Differentiating once again with respect to k yields

$$E\tilde{z}^2 u_{11}(w_0 + k\tilde{z}, \tilde{y}) = [\pi'^2_u E u_{11}(w_0 - \pi_u(k), \tilde{y}) - \pi''_u(k) E u_1(w_0 - \pi_u(k), \tilde{y}). \tag{85}$$

This implies that

$$\pi''_u(0) = -\frac{Eu_{11}(w_0, \tilde{y})}{Eu_1(w_0, \tilde{y})} E\tilde{z}^2. \tag{86}$$

Similarly, for v we have

$$\pi''_v(0) = -\frac{Ev_{11}(w_0, \tilde{y})}{Ev_1(w_0, \tilde{y})} E\tilde{z}^2. \tag{87}$$

Now π_u and π_v can be written as the forms of a Taylor expansion around $k = 0$:

$$\pi_u(k) = -\frac{1}{2} \frac{Eu_{11}(w_0, \tilde{y})}{Eu_1(w_0, \tilde{y})} E\tilde{z}^2 k^2 + o(k^2) \tag{88}$$

and

$$\pi_v(k) = -\frac{1}{2} \frac{Ev_{11}(w_0, \tilde{y})}{Ev_1(w_0, \tilde{y})} E\tilde{z}^2 k^2 + o(k^2). \tag{89}$$

¿From (82) we know that, when $k \to 0$, we get $\pi_u < \pi_v$ for $G(x, y)$ such that $G_Y(y)$ has positive support on interval $[n'_1, n'_2]$. This is a contradiction. Q.E.D.

10 References

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