Increases in Risk and Optimal Portfolio

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Abstract

We study the effect of riskiness on optimal portfolio. As discussed by Levy (1992), the main drawback of the standard model with one decision variable and one risky asset developed over the last twenty-five years, following the contributions of Rothschild and Stiglitz (1970, 1971) and Hadar and Russell (1969), is in the area of finance since this framework is not appropriate to study portfolio diversification. Our purpose is to answer the following question: How a mean preserving spread on the returns of a given asset affect the composition of an optimal portfolio with two risky assets and one riskless asset? We propose a methodology to answer this difficult question and we show that we must introduce different restrictions on the set of von Newman-Morgenstern utility functions and that of returns distribution functions to obtain intuitive results. However, we do not have to limit the analysis to the mean-variance model.

JEL : D81, G11.

Résumé

Nous étudions l'effet du risque sur un portefeuille optimal. Comme discuté par Levy (1992), la principale lacune du modèle standard avec une variable de décision et un actif risqué développé au cours des vingt-cinq dernières années, suivant les contributions de Rothschild et Stiglitz (1970, 1971) et Hadar et Russell (1969), est dans le domaine de la finance, étant donné que ce cadre d'analyse est non approprié pour étudier la diversification d'un portefeuille. Notre but est de répondre à la question suivante : comment un accroissement de risque sur les rendements d'un actif affecte-t-il la composition d'un portefeuille optimal ayant deux actifs risqués et un actif sans risque ? Nous proposons une méthodologie pour répondre à cette difficile question et nous montrons comment des restrictions sur l'ensemble des fonctions d'utilité von Newman-Morgenstern et celui des fonctions de distribution des rendements doivent être introduites pour obtenir des résultats intuitifs. Par contre, nous n'avons pas à limiter l'analyse au modèle moyenne-variance.

JEL : D81, G11.

1. Introduction

Since the contributions of Rothschild and Stiglitz (1970–1971) there has been a proliferation of articles on the effect of increases in risk on the optimal decision variables of economic problems under uncertainty (see the recent articles by Hadar and Seo (1990); Eeckhoudt and Kimball (1992); Dionne, Eeckhoudt and Gollier (1993); Meyer and Ormiston (1994); Bigelow and Menezes (1995); Gollier (1995), Dionne and Gollier (1996) and Eeckhoudt, Gollier, Schlesinger (1996)). Some papers have extended this literature by considering problems with two random parameters but were restricted to applications with only one decision variable which implies that this literature cannot yet study the effect of a general increase in risk on an optimal portfolio. Moreover, as discussed by Levy (1992) in his survey, the main drawback of the standard one decision variable model is in the area of finance, since the models cannot be used for the study of efficient diversification strategies. More recently, Meyer and Ormiston (1994) stated : "Extension of these comparative results to portfolios with more than two assets is difficult. This is because more than one decision variable and first order conditions must be analyzed" (p.611).

The object of this article is to extend significantly this literature by proposing a model with two decision variables and two dependent random variables. In the literature on optimal portfolio analysis, restrictions are often imposed on the distribution of the rates of return and/or the utility function of the decision makers. Any form of comparative statics analysis becomes very complicated when more than one risky asset is in the portfolio. Ross (1981) showed, for example, that we must restrict the Arrow–Pratt measure of risk aversion in the presence of two risky assets, if we want to obtain the intuitive result that a decision maker, with decreasing absolute risk aversion, will increase his investment in the risky asset following an increase in his initial wealth. But, as demonstrated by Machina (1982) and Epstein (1985), even the Ross' definition of risk aversion is not strong enough to make the comparative statics analysis if the increment in wealth is random instead of being non-stochastic (see also Eeckhoudt, Gollier and Schlesinger (1996)). Machina needs that the two base wealth distributions being comparable by using the criteria of first-order stochastic dominance. Epstein proposes another set of restrictions and shows that his analysis implies mean-variance utility even if his application is restricted to one decision variable. In this paper we consider a different set of restrictions by using a *ceteris paribus* assumption on changes in risk[1].

Although this form of comparative static analysis associated to the variation of wealth is not directly related to our problem, it is not without any link. It is well known that decreasing absolute risk aversion is a sufficient condition to sign the effect of an increase in initial wealth on the optimal portfolio (one random variable-one decision variable model). Decreasing risk aversion in also part of the set of sufficient conditions (although it is not necessary) to sign the effect of increases in risk of the risky asset on risk averse individuals' portfolio composition. In general, however, we need more restrictive assumptions on the utility function to sign the effect of a Rothschild–Stiglitz mean preserving spread on optimal decision variables than for an increase in base wealth.

One way that was adopted in the finance literature to simplify the analysis was to propose that risk averse individuals act as they hold the same portfolio of risky assets and only modify the composition between that portfolio and the riskless asset (two-fund separation). Cass and Stiglitz (1970) have demonstrated that such behaviour implies that individuals hold a portfolio of two assets and corresponds to specific utility functions. This approach has been intensively used over the recent years to analyze, for example, the effects of, both, increases in initial wealth and mean preserving spreads on the composition of individuals' portfolio (Hadar and Seo, 1990; Meyer and Ormiston, 1994

[1] We know from Meyer (1987) and Epstein (1985) that mean-variance (or mean-standard deviation) does not imply quadratic utility functions or normal distributions.

and Dionne and Gollier, 1996). This methodology with one decision variable[2] is not free of criticism since it cannot explain how the increase in the riskiness of some risky assets affect the composition of the risky portfolio.

Hadar and Seo (1990) assumed that the risk returns are independently distributed. They proposed conditions on preferences to obtain that, for a mean preserving spread on the rate of return of a risky asset, the proportion of the portfolio invested in that asset does not increase. Meyer (1992) and Meyer and Ormiston (1994) extended their result by showing that the condition proposed by Hadar and Seo remains necessary and sufficient (along with $U'(\cdot)$ convex) for dependent risky returns when an appropriate restriction is imposed on the definition of increase in risk (*ceteris paribus* condition). It is important to emphasize here that a *ceteris paribus* condition will play an important role in our analysis. As mentioned above, such condition was not discussed in Machina (1982) and Epstein (1985).

Dionne and Gollier (1992, 1996) proposed a different extension to Hadar and Seo (1990) contribution by considering restrictions on the set of changes in risk for all risk averse individuals instead of restrictions on utility functions. They showed that the order of Linear Stochastic Dominance (Gollier, 1995) can be extended to models with two dependent risky assets but, again, their model contains only one decision variable. They also had to impose a *ceteris paribus* condition.

Indeed, they proposed to use the joint distribution of x_1 and x_2 as $dF(x_1{*}x_2)dG(x_2)$ where $F(x_1{*}x_2)$ is the distribution of x_1 conditional on x_2 and $G(x_2)$ is the marginal distribution of x_2. In this framework the *ceteris paribus* assumption consists to assume that the marginal distribution of x_2 is unchanged when a change in risk is imposed on the conditional distribution of x_1. Meyer and Ormiston (1994) extended the analysis of Hadar and Seo

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[2] For models with two decisions variables but with one random parameter see Dionne and Eeckhoudt (1984) and Eeckhoudt, Meyer and Ormiston (1997). For standard models with one random variable and one decision variable, see Hanoch and Levy (1969), Hammond (1974), Fishburn and Porter (1976), Cheng, Magill and Shafer (1987).

(1990) by supposing that the conditional distribution of x_1 is altered in the following way : "as x_1 is changed, the marginal cumulative distribution of x_2 is assumed to be unchanged" (p.606, with appropriate modifications of notation) which is related to the definition proposed by Dionne and Gollier (1992). An example where the conditions imposed on the change of x_1 are met is the following : let $x_1^1 = x_1^0+d$ where d is a random variable which satisfies $E(d*x_1^0,x_2) = 0$ (Meyer and Ormiston, 1994). A sufficient condition to obtain the desired comparative static result is that the noise (d) added to the initial random variable x_1^0 be independent of both x_1^0 and x_2 whatever the dependence between x_1^0 and x_2.

In this article we propose a detailed analysis of a three assets portfolio with two decision variables and show how the increase in risk of one risky asset affects the composition of risk averse individuals' portfolios. In the next section, we propose a model with two random and two decision variables and present conditions that characterize an optimal portfolio. A new sufficient condition is proposed to obtain a direct relationship between the values of the decisions variables and the covariances of their respective returns. In section 3, the comparative statics in terms of increases in risk is analysed. Four examples are studied in detail. Section 4 discusses the *ceteris paribus* assumption. The last section summarizes the main results (contained in Propositions 4 and 5) and proposes some extensions.

2. A portfolio with two random variables and two decision variables

2.1 The maximization problem

The basic model with one decision variable can be extended as follows. A strictly risk averse individual must allocate his normalized initial wealth $W_0 \equiv 1$ in two risky assets and a risk free asset. Initial position is equal to

$$1 = z_0 + z_1 + z_2 \qquad (1)$$

where z_0, z_1 and z_2 are initial investments in the risk free asset Z_0 and two risky assets Z_1 and Z_2. We assume that the choice set is compact. This assumption implies that the investor has a limited access to the credit market which means that he cannot borrow + ∞.

End of period random wealth is then equal to :

$$W(z_1,z_2) = (1 + x_0) + z_1(x_1-x_0) + z_2(x_2-x_0)$$

where x_0 is the risk free rate of return and x_1 and x_2 are random rates of return for Z_1 and Z_2 respectively.

Since x_0 is a constant, $W(z_1, z_2)$ can be rewritten as $z_1(x_1-x_0) + z_2(x_2-x_0)$ without any loss of generality in order to simplify the notation. z_1* and z_2* solve the following maximization problem :

$$\underset{z_1,z_2}{\mathrm{Max}}\ E\,U\big(W(z_1,z_2)\big) = \underset{z_1,z_2}{\mathrm{Max}} \int_{\underline{x}_2}^{\overline{x}_2}\int_{\underline{x}_1}^{\overline{x}_1} U\big(z_1(x_1-x_0)+z_2(x_2-x_0)\big)\,d^2H(x_1,x_2) \qquad (2)$$

where $[\underline{x}_1,\overline{x}_1]$ and $[\underline{x}_2,\overline{x}_2]$ are respectively the support of x_1 and x_2 and $H(x_1,x_2)$ is the joint distribution of the two random rates of return. The continuity of $U(\cdot)$ and the fact that the choice set is compact insure the existence of a solution. Assuming that we limit the analysis to interior solutions, the first order conditions of the above problem are :

$$\int_{\underline{x}_2}^{\overline{x}_2}\int_{\underline{x}_1}^{\overline{x}_1} U'\big(z_1(x_1-x_0)+z_2(x_2-x_0)\big)(x_1-x_0)\,d^2H(x_1,x_2) = 0, \qquad (3)$$

$$\int_{\underline{x}_2}^{\overline{x}_2}\int_{\underline{x}_1}^{\overline{x}_1} U'\big(z_1(x_1-x_0)+z_2(x_2-x_0)\big)(x_2-x_0)\,d^2H(x_1,x_2) = 0. \tag{4}$$

The above conditions are necessary and sufficient for an optimal solution under strict risk aversion or when U is strictly concave. By application of the definition of the covariance, the two first order conditions can be written as :

$$E\,U'\big(z_1(x_1-x_0)+z_2(x_2-x_0)\big)\; m_1 + \mathrm{cov}(U'(W),x_1-x_0) = 0 \tag{5}$$

$$E\,U'\big(z_1(x_1-x_0)+z_2(x_2-x_0)\big)\; m_2 + \mathrm{cov}(U'(W),x_2-x_0) = 0 \tag{6}$$

where $m_1 = E(x_1 - x_0)$ and $m_2 = E(x_2 - x_0)$. In general, we cannot solve the above conditions to get explicit values of z_1* and z_2*. However, for our purpose, explicit solutions are not necessary. The next four examples will be useful for both motivating Propositions 1, 2 and 3, and deriving comparative statics results.

1) U is a quadratic utility function, which means that $U'''(W) = 0$. The two first order conditions become

$$z_1(\sigma_{11}+m_1^2) + z_2(\sigma_{12}+m_1m_2) = m_1 \tag{7}$$

$$z_2(\sigma_{22} + m_2^2) + z_1(\sigma_{12}+m_1m_2) = m_2 \tag{8}$$

where σ_{ii} and σ_{ij} are respectively for the variance of x_i and the covariance between x_i and x_j.

Solving the system of two equations yields the following explicit values for z_1* and z_2* :

$$z_1^* = \frac{m_1\sigma_{22} - m_2\sigma_{12}}{\left(m_1^2\sigma_{22} - \sigma_{12}^2 + \sigma_{11}\sigma_{22} + m_2^2\sigma_{11} - 2\,\sigma_{12}\,m_1 m_2\right)} \tag{9}$$

$$z_2^* = \frac{m_2\sigma_{11} - m_1\sigma_{12}}{\left(m_1^2\sigma_{22} - \sigma_{12}^2 + \sigma_{11}\sigma_{22} + m_2^2\sigma_{11} - 2\,\sigma_{12}\,m_1 m_2\right)}. \tag{10}$$

By using the fact that the determinant of the variance-covariance matrix is positive, we can show that the common denominator is strictly positive. Consequently, the optimal values are function of four different parameters (see Mossin (1973) for a detailled analysis of the different cases). To simplify both the presentation and the interpretation of the results, we will assume that $m_2 = 0$. It is clear that even if $m_2 = 0$, the asset proportion z_2 is not trivially equal to zero since it can be used for hedging purposes when x_2 is correlated with x_1. Other cases with different values of m_1 and m_2 are discussed in Section 5. When $m_2 = 0$, (9) and (10) become respectively :

$$z_1^* = \frac{m_1\sigma_{22}}{\left(\sigma_{11}\sigma_{22} - \sigma_{12}^2 + m_1^2\,\sigma_{22}\right)} \tag{9'}$$

$$z_2^* = \frac{-\,m_1\sigma_{12}}{\left(\sigma_{11}\sigma_{22} - \sigma_{12}^2 + m_1^2\sigma_{22}\right)}, \tag{10'}$$

where the common denominator is strictly positive.

When $m_1 > 0$, we verify that $z_1^* > 0$ and $z_2^* < 0$ when $\sigma_{12} > 0$ and $z_1^* > 0$ and $z_2^* > 0$ when $\sigma_{12} < 0$. An other case of interest is when $m_1 < 0$. We verify that $z_1^* < 0$ and $z_2^* > 0$ when $\sigma_{12} > 0$ and $z_1^* < 0$, $z_2^* < 0$ when $\sigma_{12} < 0$. Consequently Sign $(z_1^*\, z_2^*) = -$Sign Cov (x_1, x_2) and Sign (z_1^*)

= Sign (m_1). It is important to repeat here that since the utility function is quadratic, only the first two moments of the distribution do matter. However, as pointed out by Meyer (1987), other utility functions can be used for mean-variance analysis. Our second example is the mean-standard-deviation utility case.

2) We now suppose that the welfare of the risk averse agent is represented by $V(\mu,\sigma)$ where μ is the mean of the portfolio and σ is its standard deviation. To be more precise

$$\mu \equiv E(W(z_1,z_2)) = m_1 z_1 + m_2 z_2 \tag{11}$$

$$\sigma \equiv \left(z_1^2\sigma_{11} + z_2^2\sigma_{22} + 2\sigma_{12}z_1z_2\right)^{1/2} \tag{12}$$

We use the standard deviation in accordance to Meyer's comment (1987) that two-moment decision models correspond to a broader class of utility functions having the appropriate convexity properties. Maximizing $V(\mu,\sigma)$ over z_1 and z_2 yields as first order conditions (when $m_2 \equiv 0$) :

$$V_1 m_1 + \frac{V_2}{\sqrt{\sigma^2}}\left(z_1\sigma_{11}+z_2\sigma_{12}\right) = 0 \tag{13}$$

$$\frac{V_2}{\sqrt{\sigma^2}}\left(z_1\sigma_{12}+z_2\sigma_{22}\right) = 0 \tag{14}$$

where $V_1 > 0$ and $V_2 < 0$ are for $dV/d\mu$ and $dV/d\sigma$ respectively, which implies that

$$z_1 = \frac{-V_1\sigma}{V_2}\left(\frac{m_1\sigma_{22}}{\sigma_{22}\sigma_{11}-\sigma_{12}^2}\right) \tag{15}$$

$$z_2 = \frac{V_1\sigma}{V_2}\left(\frac{m_1\sigma_{12}}{\sigma_{22}\sigma_{11}-\sigma_{12}^2}\right). \tag{16}$$

We observe that the results are similar to those obtained in the preceding case while z_1 and z_2 are not explicit solutions. We must take into account that the inverse of the marginal rate of substitution ($-V_1/V_2$) between μ and σ is a function of both μ and σ and σ is itself function of σ_{11}, σ_{22} and σ_{12}.

Finally, when $V(\mu,\sigma^2) = \mu - a\sigma^2$ $(a > 0)$, z_1^* and z_2^* can be derived explicitly. It can be shown that this case may correspond to $U(W) = -e^{-\delta W}$ or to constant absolute risk aversion (Epstein, 1985), which introduces our third example.

3) We now assume that x_1 and x_2 are random variables that are bivariate normally distributed, which implies that $W(z_1, z_2)$ is also normally distributed. Therefore, applying the Stein's lemma[3], when $m_2 \equiv 0$, $\mathrm{cov}(U'(W(z_1, z_2)),x_2-x_0) \equiv EU''(W(z_1, z_2))\,\mathrm{cov}(z_1(x_1-x_0) + z_2(x_2-x_0), (x_2-x_0)) = 0$ which is equivalent to

$$EU''(W(z_1, z_2))(z_1\sigma_{12}+z_2\sigma_{22}) = 0 \tag{17}$$

implying that $z_2 = -z_1\frac{\sigma_{12}}{\sigma_{22}}$ from the first order condition for z_2.

[3] When x_1 and x_2 are bivariate normally distributed, we can write by using the Stein's lemma $\mathrm{cov}(g(x_1), x_2) = E(g'(x_1))\,\mathrm{cov}(x_1, x_2)$ provided that $g(x_1)$ is differentiable and meets some regulatory conditions (see Huang and Litzenberger, 1988, section 4.14 for more details).

Applying the Stein's lemma to the first order condition for z_1 yields

$$EU'(W)\,E\left(x_1 - x_0\right) = -E\left(U''(W)\right)\left(z_1\sigma_{11} + z_2\sigma_{12}\right) \tag{18}$$

where $W \equiv W(z_1, z_2)$ for the reminder of this section.

Substituting the value of z_2 from (17) in (18), we obtain:

$$z_1 = \frac{-EU'(W)}{EU''(W)}\;\frac{m_1\sigma_{22}}{\sigma_{11}\sigma_{22} - \sigma_{12}^2} \tag{19}$$

$$z_2 = \frac{EU'(W)}{EU''(W)}\;\frac{m_1\sigma_{12}}{\sigma_{11}\sigma_{22} - \sigma_{12}^2} \tag{20}$$

and different values for z_1 and z_2 can be derived for different assumptions about σ_{12} and m_1. We also observe from (19) and (20) that Sign $(z_1{*}z_2{*}) = -$ Sign cov (x_1, x_2) without any assumption on the utility function. When $V(\mu,\sigma^2) = \mu - a\sigma^2$ or when $U(W) = -e^{-\delta W}$, the corresponding values of (19) and (20) are obtained by substituting $\frac{1}{2a}$ or $\frac{1}{\delta}$ to $\frac{-EU'(W)}{EU''(W)}$.

Two important conclusions come from these examples. For $m_2 = 0$, Sign $(z_1{*})$ = Sign (m_1) and Sign $(z_1{*}z_2{*}) = -$Sign cov (x_1, x_2). The following three propositions show how these results can be obtained for all concave utility functions.

2.2 A charactherization of the optimal portfolio

Let us write $d^2 H(x_1, x_2) \equiv dF(x_1|x_2)\, dG(x_2)$ where $F(x_1|x_2)$ is the distribution of x_1 conditional on x_2 and $G(x_2)$ is the marginal distribution of x_2. In the reminder of the paper, we assumne that $F(x_1|x_2)$ is differentiable with respect to x_2 to simplify the presentation. However, this assumption is not necessary to get the results. We now propose a sufficient condition on $F(x_1|x_2)$ in order to characterize the optimal portfolio for all concave utility functions.

Proposition 1 :When $m_2=0$, suppose that $F(x_1|x_2)$ is monotone in x_2 for every x_1 then $\text{Sign}\,(z_1^*z_2^*) = -\text{Sign}\,(\text{cov}(x_1, x_2))$ for all risk averse individuals.

Proof : By the first order condition (4) we have $EU_{z_2} = \int_{\underline{x_2}}^{\overline{x_2}} (x_2-x_0)\, I(x_2)dG(x_2) = 0$

where $I(x_2) \equiv \int_{\underline{x_1}}^{\overline{x_1}} U^{/}(\cdot)\, dF(x_1|x_2)$.

Taking the first derivative of $I(x_2)$ and using the Leibniz rule, we get

$$I^{/}(x_2) = z_2 \int_{\underline{x_1}}^{\overline{x_1}} U^{//}(\cdot)\, dF\,(x_1|x_2) - z_1 \int_{\underline{x_1}}^{\overline{x_1}} U^{//}(\cdot)\, F^{/}_{x_2}(x_1|x_2)\, dx_1. \quad (21)$$

By the monotonicity of $F(x_1|x_2)$ and the concavity of U, a necessary condition to have an interior solution is that:

$$\text{Sign}\,(z_1^*z_2^*) = \text{Sign}\left(F^{/}_{x_2}(x_1|x_2)\right). \quad (22)$$

In fact, suppose that (22) is not true, then we would have $\text{Sign}\,(z_1^*z_2^*) = -\text{Sign}\left(F^{/}_{x_2}(x_1|x_2)\right)$ and one can verify that $I(x_2)$ is monotonic which cannot be true if we impose an interior solution.

Moreover, by definition, when $m_2 \equiv 0$ and by using the Leibniz rule,

$$\int_{\underline{x_1}}^{\overline{x_1}} x_1 \, dF(x_1|x_2) = \overline{x_1} - \int_{\underline{x_1}}^{\overline{x_1}} \left[\int_{\underline{x_1}}^{x_1} dF(t|x_2) \right] dx_1.$$

From the last expression and, again, the fact that $m_2=0$, we can write:

$$\text{cov}\ (x_1, x_2) = -\int_{\underline{x_2}}^{\overline{x_2}} (x_2 - x_0) \left[\int_{\underline{x_1}}^{\overline{x_1}} F(x_1|x_2)\ dx_1 \right] dG(x_2). \tag{23}$$

Under the assumption that $F(x_1|x_2)$ is monotone in x_2 for every x_1,

$$\text{Sign}\left(\text{cov}\,(x_1, x_2)\right) = -\ \text{Sign}\ \left(F'_{x_2}\,(x_1|x_2)\right). \tag{24}$$

Expressions (22) and (24) end the proof of Proposition 1. ■

We can also show the next result :

Proposition 2 : Suppose $m_2 = 0$ and x_1 and x_2 are independent random variables, then $z_2^* = 0$.

Proof : If x_1 and x_2 are independent then $dF(x_1|x_2) = dF(x_1)$. We can then write the first order condition as :

$$\begin{aligned} EU_{z_2}\left(W\left(z_1,\ 0\right)\right) &= \int_{\underline{x_2}}^{\overline{x_2}} \left(x_2 - x_0\right) g\left(x_2\right) dx_2 \int_{\underline{x_1}}^{\overline{x_1}} U'\left(z_1\left(x_1 - x_0\right)\right) dF\left(x_1\right) \\ &= m_2 \int_{\underline{x_1}}^{\overline{x_1}} U'\left(z_1\left(x_1 - x_0\right)\right) dF\ (x_1) \\ &= 0. \end{aligned}$$

Note that if we keep the same assumption on the monotonicity of $F(x_1|x_2)$, then the independence assumption in Proposition 2 can be replaced by the assumption of a nil covariance[4].

Proposition 1 shows that even if the second asset is actuarially fair ($m_2 = 0$) the asset proportion z_2^* at the optimum is not trivially equal to zero since it can be used for hedging purposes when x_2 is correlated with x_1. If the two assets are not correlated, then a risk averse investor[5] would not invest in the second asset which confirms the existence of hedging in the optimal portfolio. This conclusion in confirmed by the relation through the covariance of the two random variables given in Propositions 1 and 2.

We must now discuss on the sufficient condition that $F(x_1|x_2)$ is monotone in x_2. First, notice that this condition is met naturally when two variables are bivariate normally distributed (see the Appendix for the details).

A more general example is the following. Let's consider $\tilde{\mho}_1$ and $\tilde{\mho}_2$, two dependent random variables. We construct $\tilde{x}_1$ and $\tilde{x}_2$ as:

$$\tilde{x}_2 = a + b\tilde{\mho}_1, \text{ and } \tilde{x}_1 = c + d\tilde{\mho}_2 + e\tilde{\mho}_1 \qquad (25)$$

We can write:

$$F(x_1|x_2) \qquad = \Pr\left(\tilde{x}_1 \le x_1 | \tilde{x}_2 = x_2\right)$$

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[4]In fact, one can prove that if $m_2 = 0$ and $F(x_1|x_2)$ is monotone in x_2 for every x_1, then $\text{cov}\left(x_1, x_2\right) = 0$ implies that x_1 and x_2 are independent. We know that the reverse is always true.

[5]Even if risk aversion does not figure in the proof of Proposition 2, one should keep in mind that risk aversion makes the first order condition necessary and sufficient for a maximum. This is exactly what we use in the proof of Proposition 2.

$$= \Pr\left(\tilde{x}_1 \le x_1 | \tilde{\mho}_1 = \frac{x_2 - a}{b}\right)$$

$$= \Pr\left(d\tilde{\mho}_2 \le x_1 - c - \frac{e}{b}\left(x_2 - a\right)\right)$$

As we can see, $F\,(x_1|x_2)$ is always monotone in x_2 and the sign of this monotonicity depends on the sign of b and e, i.e.:

$$\operatorname{sing}\left(F_{x_2}\left(x_1|x_2\right)\right) = -\operatorname{sign}(be).$$

Note that if $\tilde{\mho}_1$ and $\tilde{\mho}_2$ are normal then $\left(\tilde{x}_1,\ \tilde{x}_2\right)$ is bivaritate normally distributed. But the example clearly shows that $\tilde{\mho}_1$ and $\tilde{\mho}_2$ have not to be normal to obtain the desired dependance between $\tilde{x}_1$ and $\tilde{x}_2$. The transformations $\tilde{x}_1$ and $\tilde{x}_2$ can also be power functions if $\tilde{\mho}_1$ and $\tilde{\mho}_2$ are positive values, i.e.:

$$\tilde{x}_2 = \alpha\tilde{\mho}_1^{\beta},\ \tilde{x}_1 = \gamma\tilde{\mho}_1^{\delta}\,\tilde{\mho}_2^{\epsilon}.$$

Now we turn to identify the different positions (long vs short) that the investor takes on the first risky asset depending on the expected return. We know that, in the situation where an agent is allocating his wealth between a risk-free asset and a risky asset, a necessary and a sufficient condition for an agent to invest a positive amount in the risky asset is that the expected return exceeds that of the riskless asset. In the next proposition we try to generalize this result to our model when we add another risky asset. In fact we can prove the next result :

Proposition 3 : Suppose that $m_2 = 0$, then a necessary and sufficient condition for having a positive z_1^* is that $m_1 > 0$.

Proof : We need to prove that Sign $\left(z_1^*\right)$ = Sign (m_1) or equivalently that the first order condition (3) $EU_{z_1}(W(z_1,z_2))$, evaluated at $z_1^*=0$, has the same sign as m_1. When $z_1^*=0$, we verify that (4) is reduced to

$$EU_{z_2}\left(W\left(0,z_2\right)\right) = \int_{\underline{x_2}}^{\overline{x_2}} (x_2 - x_0)\, U'(z_2 (x_2 - x_0))\, dG(x_2)$$

which implies that Sign ($EU_{z_2}(W(0,z_2))$) = −Sign (z_2).

By the above expression we see that if the individual invests 0 in the first asset then he will invest 0 in the second asset. From first order condition (3), $EU_{z_1}(W(0,0)) =$ $U'(0)m_1$. Since $U'(\cdot) > 0$, by the concavity of U we have that Sign (z_1^*) = Sign (m_1). ■

3. Comparative static analysis

3.1 General Framework

Let us consider the following comparative static problem : how a mean preserving spread of x_1 affects the composition of the optimal portfolio? This question implies that we must consider simultaneously the effect of the mean preserving spread on the two decision variables.

Suppose that we use the following notation. An increase in risk is designed by a partial derivative of the joint distribution function with respect to a parameter r, for risk. Then $H(x_1,x_2*r)$ is the joint cumulative distribution of x_1 and x_2 for a given risk r of x_1. Now in order to take into account of the *ceteris paribus* assumption we will define

$d^2H(x_1,x_2*r) \equiv dF(x_1*x_2,r)dG(x_2)$ and we will use $F''_{x_1,r}dG(x_2)dx_1dr$ for an increase in risk on x_1 under the *ceteris paribus* hypothesis.

Differentiating the two first order conditions (3) and (4) with respect to z_1,z_2 and r yields :

$$\left\{\int_{\underline{x_1}}^{\overline{x_1}}\int_{\underline{x_2}}^{\overline{x_2}} U''(\cdot)(x_1-x_0)^2 dF(x_1*x_2,r)dG(x_2)\right\} dz_1 + \left\{\int_{\underline{x_1}}^{\overline{x_1}}\int_{\underline{x_2}}^{\overline{x_2}} U''(\cdot)(x_1-x_0)(x_2-x_0)dF(x_1*x_2,r)dG(x_2)\right\} dz_2 + \int_{\underline{x_1}}^{\overline{x_1}}\int_{\underline{x_2}}^{\overline{x_2}} U'(\cdot)(x_1-x_0)F''_{x_1,r}dx_1drdG(x_2) = 0 \tag{26}$$

$$\left\{\int_{\underline{x_1}}^{\overline{x_1}}\int_{\underline{x_2}}^{\overline{x_2}} U''(\cdot)(x_1-x_0)(x_2-x_0)dF(x_1*x_2,r)dG(x_2)\right\} dz_1 + \left\{\int_{\underline{x_1}}^{\overline{x_1}}\int_{\underline{x_2}}^{\overline{x_2}} U''(\cdot)(x_2-x_0)^2 dF(x_1*x_2,r)dG(x_2)\right\} dz_2 + \int_{\underline{x_1}}^{\overline{x_1}}\int_{\underline{x_2}}^{\overline{x_2}} U'(\cdot)(x_2-x_0)F''_{x_1,r}dx_1drdG(x_2) = 0. \tag{27}$$

where $U'(\cdot)$ and $U''(\cdot)$ are written for $U'(W(z_1, z_2))$ and $U''(W(z_1, z_2))$ to save space.

Rearranging the two above relations in matrix form and applying the Cramer's rule we obtain :

$$*H*\frac{dz_1^*}{dr} = \left[\int_{\underline{x}_1}^{\overline{x}_1}\int_{\underline{x}_2}^{\overline{x}_2} U'(\cdot)(x_2-x_0)F''_{x_1,r}\,dx_1\,dG(x_2) \cdot \int_{\underline{x}_1}^{\overline{x}_1}\int_{\underline{x}_2}^{\overline{x}_2} U''(\cdot)(x_1-x_0)(x_2-x_0)\,dF(x_1*x_2,r)\,dG(x_2) \right]$$

$$- \left[\int_{\underline{x}_1}^{\overline{x}_1}\int_{\underline{x}_2}^{\overline{x}_2} U'(\cdot)(x_1-x_0)F''_{x_1,r}\,dx_1\,dG(x_2) \cdot \int_{\underline{x}_1}^{\overline{x}_1}\int_{\underline{x}_2}^{\overline{x}_2} U''(\cdot)(x_2-x_0)^2\,dF(x_1*x_2,r)\,dG(x_2) \right] \qquad (28)$$

$$*H*\frac{dz_2^*}{dr} = \left[\int_{\underline{x}_1}^{\overline{x}_1}\int_{\underline{x}_2}^{\overline{x}_2} U''(\cdot)(x_1-x_0)(x_2-x_0)\,dF(x_1*x_2,r)\,dG(x_2) \cdot \int_{\underline{x}_1}^{\overline{x}_1}\int_{\underline{x}_2}^{\overline{x}_2} U'(\cdot)(x_1-x_0)F''_{x_1,r}\,dx_1\,dG(x_2) \right]$$

$$- \left[\int_{\underline{x}_1}^{\overline{x}_1}\int_{\underline{x}_2}^{\overline{x}_2} U''(\cdot)(x_1-x_0)^2\,dF(x_1*x_2,r)\,dG(x_2) \cdot \int_{\underline{x}_1}^{\overline{x}_1}\int_{\underline{x}_2}^{\overline{x}_2} U'(\cdot)(x_2-x_0)F''_{x_1,r}\,dx_1\,dG(x_2) \right] \qquad (29)$$

where the determinant of the Hessian Matrix $\underset{2x2}{**} \equiv *H* > 0$ for a maximum. Since both conditions are symmetric, we will first focus our attention to (28). We now analyze in detail each of the four terms. To simplify the notation, let us rewrite (28) as

$$*H*\frac{dz_1^*}{dr} = \Delta_1 \cdot \Delta_2 - \Delta_3 \cdot \Delta_4 \qquad (30)$$

where

$$\Delta_1 \equiv \int_{\underline{x}_1}^{\overline{x}_1}\int_{\underline{x}_2}^{\overline{x}_2} U'(\cdot)(x_2-x_0)F''_{x_1,r}(x_1*x_2,r)\,dx_1 dG(x_2) \qquad (31)$$

$$\Delta_2 \equiv \int_{\underline{x}_1}^{\bar{x}_1} \int_{\underline{x}_2}^{\bar{x}_2} U''(\cdot)(x_1 - x_0)(x_2 - x_0) dF(x_1 * x_2, r) dG(x_2) \tag{32}$$

$$\Delta_3 \equiv \int_{\underline{x}_1}^{\bar{x}_1} \int_{\underline{x}_2}^{\bar{x}_2} U'(\cdot)(x_1 - x_0) F''_{x_1,r}(x_1 * x_2, r) dx_1 dG(x_2) \tag{33}$$

$$\Delta_4 \equiv \int_{\underline{x}_1}^{\bar{x}_1} \int_{\underline{x}_2}^{\bar{x}_2} U''(\cdot)(x_2 - x_0)^2 dF(x_1 * x_2, r) dG(x_2). \tag{34}$$

Δ_3 is the Direct increase in risk effect usually analysed in the literature with one decision variable (Dionne-Gollier, 1996, and Meyer-Ormiston, 1994) while Δ_4 is from the second order condition. Δ_1 is the Pseudo Increase in Risk Effect that may be associated with the background risk effect (Eeckhoudt-Kimball, 1992, and Eeckhoudt et al, 1996). However, here this affect is endogenous instead of being exogenous. Finally, Δ_2 has never been discussed in the literature. We name this effect the Interaction Effect. These four effects are discussed in detail in the next section.

By symmetry, $H \frac{dz_2^*}{dr} = \Delta_2 \Delta_3 - \Delta_1 \Delta_4'$ where Δ_4' has a corresponding definition from (29).

3.2 Comparative Statics Results

We now present our first important result by introducing restrictions on $U(\cdot)$ and on $F''_{x_2,r}$ for a Rothschild-Stiglitz mean preserving spread. The restrictions on $U(\cdot)$ are well known in both literatures on increase in risk for one decision variable and the analysis of background risk. The new important restriction is on $F(x_1|x_2)$. We will consider in Proposition 5 particular cases of increasing risk.

Proposition 4 : Assume $U^{////} \leq 0$ and CRRA ≤ 1. Assume also that $F''_{x_2,r} = 0$ for all x_1.

Now introduce $F(x_1|x_2, r_2)$ as a mean preserving spread of $F(x_1|x_2, r_1)$ in the sense of Rothschild and Stiglitz and suppose that $G(x_2)$ is not changed. Then for $m_2 = 0$,

a) when $m_1 > 0$

$$\frac{dz_1}{dr} < 0, \ \frac{dz_2}{dr} < (\geq)\ 0 \text{ when } z_2^* > (\leq)\ 0, \text{ and } \frac{dz_0}{dr} > 0 \text{ when } \frac{dz_2}{dr} \leq 0.$$

b) when $m_1 \leq 0$ $\quad \frac{dz_1}{dr} \geq 0, \ \frac{dz_2}{dr} < (\geq)\ 0$ when $z_2^* > (\leq)\ 0$, and

$\frac{dz_0}{dr} < 0$ when $\frac{dz_2}{dr} \geq 0$.

Proof : We have to show that :

$$\text{Sign}\,(\Delta_1\Delta_2 - \Delta_3\Delta_4) = -\,\text{Sign}\,(z_1^*) \text{ and Sign}\,(\Delta_2\Delta_3 - \Delta_1\Delta_4') = -$$

Sign (z_2^*).

Let us begin with the case $m_1 > 0$. From Proposition 3, we know that $z_1^* > 0$. We first analyze the two terms Δ_1 and Δ_2 by starting with Δ_2, the Interaction Effect. This effect links z_1^* and z_2^* via the interaction between the two random parameters. This terms is very difficult to sign because it links three random variables, x_1, x_2, and $U''(\cdot)$. Moreover, an increase in the product of $(x_1-x_0)(x_2-x_0)$ does not mean a particular variation of $W(z_1^*, z_2^*) \equiv z_1^*(x_1-x_0) + z_2^*(x_2-x_0)$ and therefore does not mean a particular variation of $U''(\cdot)$. However we can prove the following result :

Lemma 1 : When $z_1^* \neq 0$, Sign (Δ_2) = Sign $(z_1^* z_2^*)$ under constant relative risk aversion (CRRA).

Proof : Δ_2 can be rewritten as

$$\Delta_2 = \int_{\underline{x_2}}^{\overline{x_2}} (x_2 - x_0) \left\{ \int_{\underline{x_1}}^{\overline{x_1}} (x_1 - x_0) \frac{U''(\cdot)}{U'(\cdot)} U'(\cdot) \right\} dF(x_1|x_2,r)\ dG(x_2). \tag{35}$$

Under CRRA we have :

$$\left[z_1 (x_1 - x_0) + z_2 (x_2 - x_0) \right] \frac{U''(\cdot)}{U'(\cdot)} = c, \text{where c is a constant.}$$

Suppose that $z_1 \neq 0$, then we have $(x_1 - x_0) \dfrac{U''(\cdot)}{U'(\cdot)} = \dfrac{c}{z_1} - \dfrac{z_2}{z_1} (x_2 - x_0) \dfrac{U''(\cdot)}{U'(\cdot)}$.

Substituting the above expression in (35) and after simplifications we get

$$\Delta_2 = \frac{c}{z_1} \int_{\underline{x_2}}^{\overline{x_2}} \int_{\underline{x_1}}^{\overline{x_1}} (x_2 - x_0)\ U'(\cdot)\ dF(x_1|x_2,r)\ dG(x_2)$$

$$- \frac{z_2}{z_1} \int_{\underline{x_2}}^{\overline{x_2}} \int_{\underline{x_1}}^{\overline{x_1}} (x_2 - x_0)^2\ U''(\cdot)\ dF(x_1|x_2,r)\ dG(x_2).$$

The first term is nil by the first order condition associated to the choice of z_2 (equation (4)). Using the concavity of $U(\cdot)$ we have :

$$\text{Sign}(\Delta_2) = \text{Sign}(z_1^* z_2^*).$$

which concludes the proof. ■

Lemma 2 in Appendix consider the special case where $z_1^* = 0$ since it was not treated in Lemma 1. Aside from CRRA, two other cases are of interest : 1) U is quadratic; 2) x_1 and x_2 are two random variables distributed according to a bivariate normal distribution. In both cases, the third moment of the distribution has no weight. Lemma 3, in Appendix, shows that under these assumptions Sign (Δ_2) = Sign $\left(z_1^* z_2^*\right)$.

We now analyze Δ_1, the Pseudo Increase in Risk Effect which can be related to the background risk effect (Eeckhoudt and Kimball, 1992, Doherty and Shlesinger, 1983 and Eeckhoudt, Gollier and Schlesinger, 1996). But here this effect is endogenous. This term measures the effect of an increase in risk of random variable x_1 on z_1^*, via the fact that z_1^* is determined simultaneously with z_2^*. In other words, when the risk of x_1 increases, this change in the distribution of x_1 affects z_2^* which in turn affects z_1^* (since both are determined simultaneously).

By defining $\theta(x_2) \equiv \int_{\underline{x}_1}^{\overline{x}_1} U'(\cdot) F''_{x_1,r} dx_1$, Δ_1 becomes :

$$\Delta_1 = \int_{\underline{x}_2}^{\overline{x}_2} \theta(x_2)(x_2 - x_0) dG(x_2). \tag{39}$$

We show that Sign (Δ_1) = $-$Sign (z_2^*).

Lemma 4 : Suppose that $F^{//}_{r,x_2} = 0$ for all x_1, then Sign (Δ_1) = −Sign (z_2*) when $U^{////}$ < 0.

Proof : See Appendix.

We must emphasize here that the sufficient conditions to obtain our result differ from those in Eeckhoudt, Gollier and Schlesinger (1996) since the latter restricted their analysis to independent risks although they used $U^{////} < 0$. In their model $F^{//}_{r,x_2} \equiv 0$ by the assumption of independence. However, the converse is not true. It is relatively easy to construct examples where $F^{//}_{r,x_2} \equiv 0$ with dependent random variables. One example is presented in the Appendix.

We now analyse Δ_3 and Δ_4. Δ_3 is identified as the Direct Increase in Risk Effect since it corresponds to the standard term of models with one decision variable. It can be rewritten as

$$\Delta_3 = \int_{\underline{x}_1}^{\overline{x}_1} \int_{\underline{x}_2}^{\overline{x}_2} U'(\cdot)(x_1 - x_0)dS(x_1 * x_2)dG(x_2) \tag{40}$$

where $S(x_1*x_2) = F(x_1*x_2, r_2) - F(x_1*x_2, r_1)$ and r_2 is more risky than r_1 by definition.

We must extend the result of Meyer and Ormiston (1994) to obtain the sign of Δ_3 since we must consider cases where the supports of the random variables can contain negative values for both x_1 and x_2. Implicitely the support of x_2 must be positive in Meyer and Ormiston (1994) so they do not need a condition on $U^{////}(\cdot)$.

Lemma 5 in Appendix shows that sign (Δ_3) = $-$Sign (z_1), which is a well known result in the literature. A sufficient condition to obtain the concavity of $U^{/}(W(\cdot))W(\cdot)$ is that CRRA $\leq$ 1 which is an intuitive condition. This means that the sufficient conditions on U(W) discussed in the literature for models with one decision variable and one random variable (Meyer, 1992; Dionne and Gollier, 1992) are sufficient to get intuitive comparative statics results for Δ_3 when $U^{////}(\cdot) < 0$ and $F^{//}_{r,x_2} = 0$.

Δ_4 is from the second order condition and is always strictly negative under strict risk aversion. Consequently the Sign of product $-\Delta_3\Delta_4$ is equal to that of Sign (Δ_3) or to $-$Sign (z_1*) as in models with one decision variable. We have now all the ingredients to complete the proof of Proposition 4.

Indeed, we obtain from Lemma 5 that Sign (Δ_3) = $-$Sign (z_1*) < 0. Since Sign ($-\Delta_4$) is always positive, it remains to study Δ_1 and Δ_2. From Lemma 1 we have that Sign (Δ_2) = Sign (z_1*z_2*) and from Lemma 4 we verify that Sign (Δ_1) = $-$Sign (z_2*). Consequently, it is immediate to verify that Sign ($\Delta_1\Delta_2$) = $-$Sign (z_1*) and to obtain $dz_1/dr < 0$. For dz_2/dr, by symmetry, Sign ($-\Delta_1\Delta_4^{/}$) = $-$Sign (z_2*) and Sign ($\Delta_2\Delta_3$) = $-$Sign (z_2*) which completes the proof for $m_1 > 0$. For $m_1 \leq 0$, the result is obtained by applying the same analysis with the appropriate signs and by using Lemma 2 when z_1* = 0. The result for $\frac{dz_2}{dr}$ is a consequence of the fact that $z_0 + z_1 + z_2 = 1$, which completes the proof. ■

It should be noted that the sufficient conditions in Proposition 4 are standard in the literature. Up to now, we have not investigated their necessity. Such exercise would imply a non-trivial extension of the analysis since the model is much more general than those used for problems with one decision variable.

We may also use restrictions on the definition of increasing risk along with weaker conditions on $U(\cdot)$. A starting point is Lemma 6 for Sign $(-\Delta_3\Delta_4)$ where the result does not require any other restriction on $U(\cdot)$ than risk aversion. However, for Sign $(-\Delta_1\Delta_2)$, sufficient conditions are that $U^{////} < 0$ and CRRA. Consequently, we can reduce restrictions on $U(\cdot)$ by adding restrictions on increasing risk. Indeed, the measure of CRRA has not to be lower than one.

Lemma 6 : Sufficient conditions on $F^{//}_{x_1,r}$ to Sign $(\Delta_3) = -$Sign (z_1^*) for all risk averse individuals is that the change in risk is one of the following : 1) conditional strong increase in risk; 2) conditional simple increase in risk.

Proof : Direct from Dionne and Gollier (1996).

Consequently, we can show:

Proposition 5 : Assume $U^{////}(\cdot) \leq 0$ and CRRA. Assume also that $F^{//}_{x_2,r} = 0$ for all x_1.

Now suppose that $F(x_1|x_2,r)$ indergoes one of the following increases in risk: 1) a conditional Strong Increase in Risk (Meyer and Ormiston, 1985); 2) a conditional Simple Increase in Risk (Dionne and Gollier, 1992); suppose also that the marginal distribution of x_2 is unchanged. Then for $m_2 = 0$,

a) when $m_1 > 0$

$$\frac{dz_1}{dr} < 0, \quad \frac{dz_2}{dr} < (\geq)\ 0 \text{ when } z_2^* > (\leq)\ 0, \text{ and } \frac{dz_0}{dr} > 0 \text{ when } \frac{dz_2}{dr} \leq 0.$$

b) when $m_1 \leq 0$

$$\frac{dz_1}{dr} \geq 0, \quad \frac{dz_2}{dr} < (\geq)\ 0 \text{ when } z_2^* > (\leq)\ 0, \text{ and } \frac{dz_0}{dr} < 0 \text{ when } \frac{dz_2}{dr} \geq 0.$$

Proof : Same as for Proposition 4 by using Lemma 6 instead of Lemma 5. ■

Notice that both propositions require strict alternance of derivatives as for proper risk aversion (Pratt and Zeckhauser, 1987), risk vulnerability (Gollier and Pratt, 1996) and proper risk behavior (Dionne, Eeckhoudt and Godfroid, 1997). Implicitly, we assume in both propositions that $U^{///}(\cdot) > 0$ since CRRA implies decreasing absolute risk aversion.

3.3 Examples

When the distribution is restricted to be a bivariate normal distribution, the results corresponding to Propositions 4 and 5 are obtained directly by differentiating (19) and (20) with respect to σ_{11} under the *ceteris paribus* condition and CARA. Notice however that we need only constant absolute risk aversion (CARA) to obtain the desired result. Consequently:

Proposition 6: When the joint distribution is bivariate normal, under the *ceteris paribus* assumption, a sufficient condition to obtain Sign $(dz_1^*/dr) = -\text{Sign}\,(z_1^*)$ and $\text{Sign}\,(dz_2^*/dr) = -\text{Sign}\,(z_2^*)$ is CARA.

Proof : By differentiating (19) and (20) under CARA and by considering the differents cases for m_1.

We now analyze the case of the quadratic utility function. Here again the analysis is direct since we have explicit values of z_1* and z_2* at the optimum.

Proposition 7 : When U(W) is quadratic, under the *ceteris paribus* assumption, $\text{Sign}(dz_1^*/dr) = -\text{Sign}(z_1^*)$ and $\text{Sign}(dz_2^*/dr) = -\text{Sign}(z_2^*)$.

The proof follows directly by differentiating (9') and (10') with respect to σ_{11} under the ceteris paribus assumption.

Notice that this result is obtained whatever both the nature of the initial distribution and the definition of increase in risk used, since all of them increase σ_{11} without affecting σ_{12} under the *cetaris paribus* assumption. This means that all the definitions of increase in risk with two random parameters used in Dionne and Gollier (1996) apply here. The role of the quadratic utility function is to set the Sign of the Pseudo Effect (Δ_1) at zero and to limit the analysis to the Direct Effect.

We now study the mean variance approach. As shown by Epstein (1985), when U(W) is exponential, $V(\mu,\sigma^2) = \mu - a\sigma^2$ implies positive linear indifference curves in the (μ,σ^2) space. Consequently,

Proposition 8 : In the mean-variance model $\text{Sign}(dz_1^*/dr) = -\text{Sign}(z_1^*)$ and $\text{Sign}(dz_2^*/dr) = -\text{Sign}(z_2^*)$, under the *ceteris paribus* assumption.

The proof is similar to that of Proposition 7 by substituting $\frac{1}{2a}$ to $\frac{1}{\delta}$.

Turning now to the mean-standard deviation space, matters are more complicated. But we can show the following result :

Proposition 9 : Suppose that the agent utility function is $V(\mu,\sigma)$ where μ and σ measure the mean and the standard deviation of the portfolio respectively. Then $\text{Sign}\,(dz_1^*/dr) = -\text{Sign}\,(z_1^*)$ and $\text{Sign}\,(dz_2^*/dr) = -\text{Sign}\,(z_2^*)$, under the *ceteris paribus* assumption.

Proof : See Appendix.

Despite their differences, Propositions 4 to 9 share one common feature : they all lead to the same comparative statics results (Sign(dz_1*/dr) = – Sign (z_1*)) and Sign (dz_2*/dr) = –Sign (z_2*). This suggests a relationship between the expected utility, the mean variance and the mean standard deviation frameworks.

4. *Ceteris paribus* assumption and covariance

We must now discuss the *ceteris paribus* assumption. From Meyer (1992), Meyer and Ormiston (1994) and Dionne and Gollier (1992, 1996), we know that such assumption permits to identify a class of distribution functions that isolate the effect of a mean preserving spread on the optimal decision variables. Gagnon (1995) showed that this assumption implies that the covariance (σ_{12}) between the random variables is maintained constant.

As an illustration, we provide an example. Suppose that the random variables x_1 and x_2 have the following realizations in a situation with two states of the world :

Initial situation (less risky)			
	X_{1i}	S_1	S_2
X_{2i}		20	40
S_1	10	0.3	0.1
S_2	30	0.2	0.4
Final situation (more risky)			
	X_{1i}	S_1	S_2
X_{2i}		15	45
S_1	10	0.2666	0.1333
S_2	30	0.2333	0.3666

Table 1

Each entry is a joint probability. The marginal probability $f(x_{1i})$ is the sum of each entry in the column S_i and the marginal probability $g(x_{2i})$ is the sum of each entry in the row S_j. The conditional probability is illustrated as follows :

$$f(X_1 = 20 | X_2 = 10) = \frac{f(X_1 = 20,\ X_2 = 10)}{f(X_2 = 10)} = \frac{0.3}{0.4} = 0.75$$

We can verify that the means of X_1 and X_2, and their conditional means, are identical under both risky situations. The covariance remains unchanged with a value of 40 as expected under the *ceteris paribus* assumption. Moreover, the *ceretis paribus* assumption is verified because the cumulative distribution of X_2 is unchanged. Only the variance of X_1 increases from 50 to 112.50. Finally, one can verify that for every value of X_2, the random variable X_1 undergoes an increase in risk as defined by Rothschild and Stiglitz (1970).

5. Extensions and conclusions

This article has proposed a framework to extend the analysis of increasing risk to models with two decision variables and two dependent random parameters. This extension permitted the comparative statics analysis of standard optimal portfolio with two random assets and one safe asset. We have proposed general conditions on the set of vNM utility functions and on the set of distribution functions to obtain intuitive comparative statics results. Suprisingly, when appropriate relationships are well identified between the random parameters, the restrictions are not more stringent than those in models with one decision variable with two dependent random parameters. However we need restrictions on both the utility function and the returns distributions. The separation of conditions either on utility or on distributions was not obtained even with the presence of a safe asset which is contrary to the two-fund separation theorem. A similar conclusion was derived by Gouriéroux and Monfort (1997) in the study of the econometrics of efficient frontiers and by Dachraoui and Dionne (1998) in the analysis of a first order shift on an optimal portfolio.

Many extensions of this contribution are possible. One would be to find conditions on changes in risk that involve less restrictions on U(·) to sign both the pseudo increase in risk effect (or the background risk effect) and the interaction effect. Kimball (1993) did a first step in that direction for the background risk effect by showing how a patently riskier change in background risk may yield the desired result on the demand for a risky asset but his model was limited to one decision variable (see also Gollier and Schlee, 1997).

Another extension would be to consider different assumptions about $m_2 = E(x_2 - x_0)$. In our analysis, the value of m_2 was constrained to be nil. To see how this type of extension is not trivial, consider again the quadratic example. When $m_2 \neq 0$, the first order conditions are given by (9) and (10). Differentiating these two conditions with respect to σ_{11} under the *ceteris paribus* assumption yields :

$$\frac{dz_1^*}{d\sigma_{11}} = -\text{Sign }(z_1^*) \text{ and is independent of } m_2$$

$$\frac{dz_2^*}{d\sigma_{11}} = \frac{m_2 m_1^2 \sigma_{22} + m_1 \sigma_{22}\sigma_{12} - m_2\,\sigma_{12}^2 - m_2^2 m_1 \sigma_{12}}{(m_1^2\sigma_{22} - \sigma_{12}^2 + \sigma_{11}\sigma_{12} + m_1^2\sigma_{12} - 2\sigma_{12}\,m_1 m_2)^2}$$

$$= \frac{(m_1 m_2 + \sigma_{12})(m_1\sigma_{22} - m_2\sigma_{12})}{(m_1^2\sigma_{22} - \sigma_{12}^2 + \sigma_{11}\sigma_{12} + m_2^2\sigma_{12} - 2\sigma_{12}\,m_1 m_2)^2} \; .$$

It is easy to see that the last expression is a function of both m_1 and m_2 even if a simple mean preserving speed is applied to the portfolio. Additional assumptions on the relative magnitudes of m_1 and m_2 would be necessary to yield intuitive results.

A third extension would be to consider n random assets instead of two. This extension would be tractable if appropriate assumptions are made on both the different covariance relationships and the respective expected values. It would be also of interest to know how the recent extension of the Rothschild-Stiglitz model made by Machina and Pratt (1997) would extend the results. In particular, how an increase in risk on x_1 would affect an optimal portfolio when the initial cantor distribution of x_1 has no mass points nor a density. Finally, non-expected utility models may also be analysed with respect to this more general portfolio model. The new tools reviewed in Chateauneuf et al. (1997) seem to be a natural starting point. See also Levy and Wiener (1998).

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APPENDIX

Bivariate normal distribution and monotoricity of $F(x_1|x_2)$

Let's consider the bivariate normal distribution:

$$f(x_1,x_2) = \frac{1}{2\pi\sigma_{11}\sigma_{22}\sqrt{1-\rho^2}}\, e^{-\frac{q}{2}},\ -\infty < x_1,\ x_2 < \infty$$

where:

$$\sigma_{11} > 0,\ \sigma_{22} > 0,\ -1 < e < 1$$

and

$$q = \frac{1}{1-\rho^2}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - 2\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right) + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right].$$

The conditional p.d.f. of $\tilde{x}_1$ given $\tilde{x}_2 = x_2$, is itself normal with mean

$$\mu = \mu_1 + \rho\frac{\sigma_1}{\sigma_2}(x_2 - \mu_2)$$

and variance

$$\sigma = \sigma_1(1-\rho^2).$$

Thus, with a bivariate normal distribution, the conditional distribution function of $\tilde{x}_1$ given $\tilde{x}_2 = x_2$ is given by

$$F\left(x_1|x_2\right) = \frac{1}{\sqrt{2\pi\sigma}} \int_{-\infty}^{x_1} e^{-½\left(\frac{u-\mu}{\sigma}\right)^2} du.$$

Taking the derivative with respect to x_2 gives

$$F_{x_2}\left(x_1|x_2\right) = \frac{1}{\sqrt{2\pi\sigma}} \int_{-\infty}^{x_1} \frac{(u-\mu)}{\sigma^2} e^{-½\left(\frac{u-\mu}{\sigma}\right)^2} du \frac{\mu}{x_2}.$$

Simple calculus show that

$$\frac{1}{\sqrt{2\pi\sigma}} \int_{-\infty}^{x_1} \frac{(u-\mu)}{\sigma^2} e^{-½\left(\frac{u-\mu}{\sigma}\right)^2} du \leq 0,$$

and

$$\frac{\mu}{x_2} = \rho \frac{\sigma_{11}}{\sigma_{22}}.$$

Then we have

$$\text{Sign}\left(F_{x_2}\left(x_1|x_2\right)\right) = -\text{Sign}\ (\rho)$$

As we can see, the monotonicity of the conditional distribution function is verified and this monotonicity is determined by the sign of the correlation between $\tilde{x}_1$ and $\tilde{x}_2$.

■

Lemma 2 : If $(z_1^* z_2^*) = 0$ and $(z_1^*, z_2^*) \neq (0,0)$, then under CRRA we have that $\Delta_2 = 0$.

Proof : Without loss of generality we can suppose that $z_1 = 0$. Under the CRRA assumption we can write Δ_2 as

$$\Delta_2 = \frac{c}{z_2} \int_{\underline{x}_2}^{\overline{x}_2} \int_{\underline{x}_1}^{\overline{x}_1} (x_1 - x_0)\; U'(\cdot)\; dF(x_1/x_2,r)\; dG(x_2).$$

The above equation is nil by the first order condition (3). The case $(z_1,z_2) = (0,0)$ is not of particular interest and can be analyzed easily. ■

Lemma 3 : If x_1 and x_2 follow a bivariate normal distribution or if U is quadratic, then Sign (Δ_2) = Sign$\left(z_1^*, z_2^*\right)$.

Proof : When the utility function is quadratic, $U'''(\cdot) = 0$ which implies that $U''(\cdot)$ is constant. Therefore

$$\Delta_2 = \int_{\underline{x}_1}^{\overline{x}_1} \int_{\underline{x}_2}^{\overline{x}_2} U''(\cdot)\left(x_1-x_0\right)\left(x_2-x_0\right)dF\left(x_1{*}x_2,r\right)dG\left(x_2\right) = U''(\cdot) \int_{\underline{x}_1}^{\overline{x}_1} \int_{\underline{x}_2}^{\overline{x}_2} \left(x_1-x_0\right)\left(x_2-x_0\right)dF\left(x_1{*}x_2,r\right)dG\left(x_2\right)$$

.

Using the definition of the covariance, the right hand side of the above equation can be written as

$$U''(\cdot)m_1m_2 + U''(\cdot)\; \mathrm{cov}(x_1-x_0,\, x_2-x_0) \tag{36}$$

which, under the assumption that $m_2 = 0$, is equal to $U''(\cdot)$ cov(x_1, x_2) since x_0 is a constant. Consequently, with the quadratic utility function, the Interaction Effect term (Δ_2) has a Sign equal to (−Sign cov(x_1, x_2)) wich is equal to Sign $(z_1{*}z_2{*})$.

We may also assume that x_1 and x_2 follow a bivariate normal distribution and obtain the same result. Let us use as a starting point the first order condition for $z_2{*}$. By symmetry the same result can be obtained from the other first order condition. As already discussed, (6) can be rewritten by using the Stein Lemma when $m_2 \equiv 0$ as :

$$EU''(\cdot)\left(z_1\sigma_{12} + z_2\sigma_{22}\right) = 0 \tag{37}$$

which is (17). Differentiating this expression with respect to z_1 yields Δ_2 :

$$EU''(\cdot)\sigma_{12} + E\left[\left(U'''(\cdot)\right)\left(x_1 - x_0\right)\right]\left[z_1\sigma_{12} + z_2\sigma_{22}\right] \tag{38}$$

which is reduced to $EU''(\cdot)\sigma_{12}$ since $[z_1\sigma_{12}+z_2\sigma_{22}] = 0$ from the first order condition (37). Therefore, under the assumption that the two random variables follow a bivariate normal distribution, we also obtain that the Sign of Δ_2 is equal to that of
$(-\text{Sign cov }(x_1, x_2))$ or to Sign $(z_1^* z_2^*)$. ■

<u>**Proof of Lemma 4**</u>: By definition of an increase in risk, $\theta(x_2)$ can be rewritten as

$$\theta(x_2) = \int_{\underline{x}_1}^{\overline{x}_1} U'(\cdot)\, dS(x_1 * x_2).$$

Integrating by parts and applying the Leibnitz rule

$$\theta(x_2) = U'(\cdot)\, S(x_1 * x_2)\Big|_{\underline{x}_1}^{\overline{x}_1} * - \int_{\underline{x}_1}^{\overline{x}_1} U''(\cdot)\, z_1 S(x_1 * x_2)\, dx_1$$

or

$$\theta(x_2) = -\int_{\underline{x}_1}^{\overline{x}_1} U''(\cdot)\, z_1 \left(d\int_{\underline{x}_1}^{x_1} S(u * x_2)\, du \right)$$

since a conditional increase in risk requires that

$$S\left(\underline{x}_1 * x_2\right) = S\left(\overline{x}_1 * x_2\right) = 0.$$

Integrating by parts again

$$\theta\left(x_2\right) = -\left\{ U''(\cdot) z_1 \int_{\underline{x}_1}^{x_1} S\left(u * x_2\right) du \, \Big*_{\underline{x}_1}^{\overline{x}_1} - \int_{\underline{x}_1}^{\overline{x}_1} U'''(\cdot) z_1^2 \left(\int_{\underline{x}_1}^{x_1} S\left(u * x_2\right) du \right) dx_1 \right\}$$

or

$$\theta\left(x_2\right) = \int_{\underline{x}_1}^{\overline{x}_1} U'''(\cdot) z_1^2 T\left(x_1 * x_2\right) dx_1 \tag{41}$$

where

$$T\left(x_1 * x_2\right) = \int_{\underline{x}_1}^{x_1} S\left(u * x_2\right) du > 0$$

by the integral definition of a mean preserving spread (Rothschild and Stiglitz, 1970)[6].

Note that $T'_{x_2}(x_1|x_2) \equiv \int_{\underline{x}_1}^{x_1} F''_{r,x_2}(u|x_2)\, du.$

Differentiating (41) with respect to x_2 yields

$$\theta'(x_2) = z_1^2 \int_{\underline{x1}}^{\overline{x1}} \left[U''''(\cdot)\ z_2 T(x_1|x_2)\ dx_1 + U'''(\cdot) T'_{x_2}(x_1|x_2) \right] dx_1. \tag{42}$$

---

[6] When $U(\cdot)$ is quadratic, $\theta(x_2) = 0$ and the Pseudo increase in risk is nul. When $U'''(\cdot) > 0$, $\theta(x_2)$ is positive.

Finally, since by assumption $T^{/}_{x_2}(x_1|x_2) = 0$ and $U^{////} < 0$, by (42) Sign $(\theta^{/}(x_2)) = -$Sign (z_2^*) which implies that $\theta(x_2)$ is monotone. Consequently,

$$\int_{\underline{x_2}}^{\overline{x_2}} \theta(x_2)\,(x_2 - x_0)\,dG(x_2) > 0\ (< 0) \text{ when } \theta'(x_2) > 0\ (< 0). \qquad \blacksquare$$

<u>An example</u> of $F^{//}_{x_2,r}$ = 0 without independance of the two random variables.

Let us suppose that two discrete distributions of x_1 conditional on two values of x_2 = 2,4 are as follows:

x_1 (x_2 = 2)	p (x_1 (x_2 = 2))	x_1 (x_2 = 4)	p (x_1 (x_2 = 4))
−4	0,09	−4	0,09
+4	0,30	+4	0,30
+10	0,40	+12	0,40
+18	0,21	+18	0,21

Both have different moments. Now assume that we introduce the same white noise structure in both distributions. As already discussed in the example presented in the introduction, the *ceteris paribus* assumption implies that E $(d|x_1, x_2) = 0$. Let us now introduce the following white noises: replace in both distribution +4 by the random variable:

+3 with probability ½

+5 with probability ½

and replace the random variable +18 by the random variable:

+14 with probability ⅓

+20 with probability ⅔.

Clearly, the expected values of the two initial conditional distributions do not change and both are more risky. However, the structure of increase in risk is independent of x_2 in the sence that both increases in risk are identical. We can verify that $F^{//}_{x_2,r} = 0$ or $T^{/}_{x_2}(x_1|x_2) = 0$ in this example.

Lemma 5 : Assume that $U^{////} \leq 0$. Assume also that $F^{//}_{x_2,r} = 0$ for all x_1. Now introduce $F(x_1{*}x_2,r_2)$ as a mean preserving spread of $F(x_1{*}x_2,r_1)$ in the sence of Rothschild and Stiglitz and suppose that $G(x_2)$ is not changed. Then Sign $(\Delta_3) = -$Sign $(z_1{*})$ if $U^{/}(W(z_1)).\ (W(z_1))$ is concave in W.

Proof : By a double integration by parts of (43) with respect to x_1, we can write Δ_3 as:

$$\Delta_3 = \int_{\underline{x_2}}^{\overline{x_2}} z_1 \left\{ \int_{\underline{x_1}}^{\overline{x_1}} \left[2U^{//}(\cdot) + WU^{///}(\cdot)\right] T(x_1|x_2)\, dx_1 \right.$$

$$\left. - \int_{\underline{x_1}}^{\overline{x_1}} z_2 (x_2 - x_0)\, U^{///}(\cdot)\, T(x_1|x_2)\, dx_1 \right\} dG(x_2)$$

or

$$z_1 \left\{ \int_{\underline{x_2}}^{\overline{x_2}} \left[\int_{\underline{x_1}}^{\overline{x_1}} [2U^{\prime\prime}(\cdot) + WU^{\prime\prime\prime}(\cdot)]\, T(x_1|x_2)\, dx_1 \right] dG(x_2) \right.$$

$$\left. - z_2 \int_{\underline{x_2}}^{\overline{x_2}} (x_2 - x_0) \left[\int_{\underline{x_1}}^{\overline{x_1}} U^{\prime\prime\prime}(\cdot)\, T(x_1|x_2)\, dx_1 \right] dG(x_2) \right\}$$

Since here, contrarily to Meyer and Ormiston (1994), both z_2 and $(x_2\text{-}x_0)$ can be either positive or negative, one cannot sign directly the above expression by using only the fact that $U^{\prime}(W)W$ is concave and $U^{\prime\prime\prime}(\cdot)$ is positive. Integrating the second term by parts with respect to x_2 one obtains for the above expression :

$$z_1 \left\{ \int_{\underline{x_2}}^{\overline{x_2}} \left[\int_{\underline{x_1}}^{\overline{x_1}} [2U^{\prime\prime}(\cdot) + WU^{\prime\prime\prime}(\cdot)]\, T(x_1|x_2)\, dx_1 \right] dG(x_2) \right.$$

$$\left. - z_2^2 \int_{\underline{x_2}}^{\overline{x_2}} \left[\int_{\underline{x_2}}^{x_2} (u - x_0)\, dG(u) \right] \left[\int_{\underline{x_1}}^{\overline{x_1}} U^{\prime\prime\prime\prime}(\cdot)\, T(x_1|x_2)\, dx_1 \right] dx_2 \right\}$$

which has a sign opposite to that of z_1 under the condition of the lemma. ■

Proof of Proposition 9: The maximization of V (μ, σ) yields (13) and (14) as first order conditions.

Taking the total differentiation of (13) with respect to σ_{11} gives after some simplifications:

$$\left\{ m_1^2 V_{11} + 2m_1 \frac{z_1^* D}{\sigma_{22}\sigma} V_{12} + \left(\frac{z_1^* D}{\sigma_{22}\sigma} \right)^2 V_{22} + \frac{V_2 D}{\sigma_{22}\sigma} \left(1 - \frac{z_1^{*2} D}{\sigma_{22}\sigma^2} \right) \right\} \frac{dz_1^*}{d\sigma_{11}} + \frac{V_2}{\sigma} \left(1 - \frac{1}{2} \frac{z_1^{*2} D}{\sigma_{22}\sigma^2} \right) z_1^* = 0. \tag{A1}$$

From (14) we can show that:

$$\frac{z_1^{*2} D}{\sigma_{22}\sigma^2} = 1. \tag{A2}$$

Substituting (A2) in (A1) yields:

$$\left[m_1^2 V_{11} + 2m_1 \frac{z_1^* D}{\sigma_{22}\sigma} V_{12} + \left(\frac{z_1^* D}{\sigma_{22}\sigma} \right)^2 V_{22} \right] \frac{dz_1^*}{d\sigma_{11}} + \frac{1}{2} \frac{V_2}{\sigma} z_1^* = 0. \tag{A3}$$

Since the Hessian matrix corresponding to V (μ, σ) is negative definite, we have in particular:

$$\left(m_1, \frac{z_1^* D}{\sigma_{22}\sigma} \right) \begin{pmatrix} V_{11} & V_{12} \\ V_{12} & V_{22} \end{pmatrix} \begin{pmatrix} m_1 \\ \dfrac{z_1^* D}{\sigma_{22}\sigma} \end{pmatrix} < 0,$$

which is equivalent to:

$$\left[m_1^2 \, V_{11} + 2m_1 \, \frac{z_1^* D}{\sigma_{22}\, \sigma} \, V_{12} + \left(\frac{z_1^* D}{\sigma_{22}\, \sigma} \right)^2 V_{22} \right] < 0.$$

The last inequality and the fact that $V_2 < 0$ imply from (A3) that:

$$\text{Sign}\left(\frac{dz_1^*}{d\sigma_{11}} \right) = -\text{Sign}\ (z_1^*).$$

From (14) we have, under the *ceteris paribus* assumption, that:

$$z_1^* = -\frac{\sigma_{22}}{\sigma_{12}} \, z_2^* \ \text{and} \ \frac{dz_1^*}{d\sigma_{11}} = -\frac{\sigma_{22}}{\sigma_{12}} \, \frac{dz_2^*}{d\sigma_{11}}.$$

Substituting these two expressions in (A3) and using the same analysis gives:

$$\text{Sign}\left(\frac{dz_2^*}{d\sigma_{11}} \right) = -\text{Sign}\ (z_2^*)$$

which completes the proof. ■