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Abstract

In this paper we show how a shift in a return distribution affects the composition of an optimal portfolio in the case of one riskless asset and two risky assets. We obtain that, in general, such a shift modifies the composition of the mutual fund. We also show that the separating conditions presented in the finance literature for the setting of the optimal portfolios, are not robust to the comparative statics following distributional shifts if we want to obtain intuitive results. This conclusion contrasts with that of Mitchell and Douglass (1997) who limited their analysis to portfolios with risky assets. Our discussion applies to a first order shift (FSD) but the same result can be obtained for increases in risk.

JEL classification: D80.

Résumé

Dans cette recherche, nous montrons comment un déplacement de premier ordre de la distribution des rendements affecte la composition d'un portefeuille optimal composé d'un actif sans risque et de deux actifs risqués. Nous obtenons que ce type de déplacement modifie la composition du fonds mutuel. Nous montrons également que les conditions de séparation présentées dans la littérature pour l'établissement d'un portefeuille optimal ne sont pas robustes à la statique comparative si nous voulons obtenir des résultats intuitifs. Cette conclusion contraste avec celle de Mitchell et Douglass (1997), qui ont limité leur analyse à des portefeuilles composés d'actifs risqués. Nos résultats peuvent être étendus directement aux accroissements de risque.

Classification JEL : D80.

1 Introduction

In the literature, recent contributions on portfolio choice and its response to distribution shifts dealt with different situations: one riskless asset-one risky asset (Rothschild and Stiglitz, 1971, Dionne et al., 1993), two risky assets (Hadar and Seo, 1990, Meyer and Ormiston, 1994, Dionne and Gollier, 1996), one riskless asset-two risky assets (Dionne et al., 1997) and, recently, an arbitrary number of assets (Mitchell and Douglass, 1997). This last contribution, however, relies on the stability of the mutual-fund separation. Here we show that such stability is not always possible and we propose a general result to mutual-fund variation following a first order stochastic dominance when the portfolio contains a safe asset.

In Mitchell and Douglass [1997], the problem is the following: an agent is allocating his initial wealth among n-risky assets: $\tilde{x}_i$, $i = 1, \ldots, n$. They show that there exists $\phi_1, \ldots, \phi_{n-1}$ and $\psi_2, \ldots, \psi_{n-1}$ and two funds $\tilde{y}_1$ and $\tilde{y}_2$ such that

$$\tilde{y}_1 = \phi_1\tilde{x}_1 + \phi_2\tilde{x}_2 + \ldots + \phi_{n-1}\tilde{x}_{n-1} + \left(1 - \phi_1 - \phi_2 - \ldots - \phi_{n-1}\right)\tilde{x}_n,$$

$$\tilde{y}_2 = \psi_2\tilde{x}_2 + \ldots + \psi_{n-1}\tilde{x}_{n-1} + \left(1 - \psi_2 - \ldots - \psi_{n-1}\right)\tilde{x}_n,$$

where the n-assets problem can be reduced to a two-fund problem. Under their assumption of mutual fund stability (following a distributional shift), one can verify easily that the solution would yield the following identities:

$$\alpha_j(r) = \frac{\phi_j - \psi_j}{\phi_1}\alpha_1(r) + \psi_j \quad \text{for } j = 2, \ldots, n, \tag{1}$$

where α_j is the amount invested in asset $\tilde{x}_j$ and r is a shift parameter. Note that parameters ϕ_j and ψ_j are independent of r. These necessary conditions are valid when the utility function is quadratic or when the returns are normally distributed and the utility function is exponential. (See Appendix for these two examples.). However the above conditions are not necessarily verified for all utility functions that are in the class permitting two-fund separation (Cass and Stiglitz, 1970). In the next section we show that for CRRA, (1) does not hold when the portfolio contains a safe asset. In fact, we obtain that $\alpha_2(r) = h(r)\alpha_1(r)$. Moreover, the necessary conditions in (1), when they hold, are not sufficient to extend the theorem of Meyer and Ormiston [1994] when all ψ_j are not restricted to be positive. This is true since $\tilde{y}_2$ is restricted to be positive in Meyer and Ormiston article. Such considerations were not taken into account explicitly in Mitchell and Douglass [1997].

2 A general result: the case of one risk free asset and two risky assets

In this section we show that the stability assumption is strong in the case of two risky assets-one risk free asset. In other words, the ratio of the two risky assets can be affected by a first order shift which means that the composition of the risky portfolio can be modified contrarily to the result in Mitchell and Douglass.

We consider a risk averse agent who allocates his wealth (normalized to one) between one risk free asset (with return x_0) and two risky assets with returns $\tilde{x}_i$ for $i = 1, 2$. We denote the cumulative distribution on asset $\tilde{x}_1$ as $F(x_1/r)$, and the cumulative distribution of the returns on asset $\tilde{x}_2$ as $G(x_2)$. For ease of presentation we suppose that $F(x_1/r)$ and $G(x_2)$ have density function given respectively by $f(x_1/r)$ and $g(x_2)$ and that the derivative of $f(x_1/r)$ with respect to r exists. The portfolio share of asset $\tilde{x}_i$ is α_i. Here we deal only with the case where $\alpha_i \geq 0$, $i = 1, 2$ and we assume that $\tilde{x}_1$ and $\tilde{x}_2$ are independent random variables. The agent's end of period wealth W is then equal to

$$W = 1 + x_0 + \alpha_1(x_1 - x_0) + \alpha_2(x_2 - x_0),$$

by using the fact that $1 = \alpha_0 + \alpha_1 + \alpha_2$.

From now on we write W as $\alpha_1(x_1 - x_0) + \alpha_2(x_2 - x_0)$. This will not result on any loss of generality since $1 + x_0$ is constant. Optimal portfolio solves the following program (P):

$$\max_{\alpha_1,\alpha_2} \int_{\underline{x}_1}^{\overline{x}_1}\int_{\underline{x}_2}^{\overline{x}_2} u(\alpha_1(x_1 - x_0) + \alpha_2(x_2 - x_0))\, dF(x_1/r)\, dG(x_2)$$

where $[\underline{x}_1, \overline{x}_1]$ and $[\underline{x}_2, \overline{x}_2]$ are respectively the support of $\tilde{x}_1$ and $\tilde{x}_2$.

Assume we have interior solutions, the first order conditions of the above problem are:

$$\int_{\underline{x}_1}^{\overline{x}_1}\int_{\underline{x}_2}^{\overline{x}_2} (x_1 - x_0)\, u'(\alpha_1(x_1 - x_0) + \alpha_2(x_2 - x_0))\, dF(x_1/r)\, dG(x_2) = 0; \tag{2}$$

$$\int_{\underline{x}_1}^{\overline{x}_1}\int_{\underline{x}_2}^{\overline{x}_2} (x_2 - x_0)\, u'(\alpha_1(x_1 - x_0) + \alpha_2(x_2 - x_0))\, dF(x_1/r)\, dG(x_2) = 0. \tag{3}$$

In particular, if the mutual-fund separation applies then the ratio $\frac{\alpha_2}{\alpha_1}$ is independent of the agent risk aversion and the mutual-fund has weights $\frac{\alpha_1}{\alpha_1+\alpha_2}$ and $\frac{\alpha_2}{\alpha_1+\alpha_2}$ on $\tilde{x}_1$ and $\tilde{x}_2$ respectively.

Definition 1 Let I be an open set in $\Re$. We say that $\{f(\cdot/r)\}_{r\in I}$ verifies the monotone likelihood ratio property (MLRP) if $\frac{f_r(x_1/r)}{f(x_1/r)}$ is decreasing in x_1 for all $r \in I$.

The MLRP is a special case of first order stochastic dominance (FSD). See Eeckhoudt and Gollier [1995] for details.

We have the next result.

Theorem 1 *Assume that (a) the utility function is CRRA; and (b) $\{f(\cdot/r)\}_{r\in I}$ verifies the MLRP condition. Let $\alpha_1^*(r)$ and $\alpha_2^*(r)$ represent optimal investment decisions in the risky fund for a given level r. Then $\frac{\alpha_2^*}{\alpha_1^*}(r)$ is increasing in r.*

Theorem 1 shows that a FSD contraction that affects one asset will reduce the weight of this asset in the optimal fund. This FSD may reduce both α_1^* and α_2^* but the relative effect on α_1^* is more important. It should be notified that $\frac{\alpha_2^*}{\alpha_1^*}$ is increasing in r for all $u(\cdot)$ that are CRRA and whatever the level of risk aversion. This means that the two-fund separation theorem holds for all r since CRRA functions are in the class of utility functions that permit mutual-fund separation. The additional restriction on MLRP is to yield a particular direction on the variation of the ratio $\frac{\alpha_2^*}{\alpha_1^*}$. Consequently, when the two-fund conditions hold, following a FSD shift, the investor must first evaluate the variations in the proportions of the risky asset and then decide how to divide his total wealth between risky and safe assets.

Proof of Theorem 1.

Differentiating the first order condition (2) with respect to r yields:

$$\begin{aligned}
&\int_{\underline{x}_1}^{\overline{x}_1}\int_{\underline{x}_2}^{\overline{x}_2}(x_1-x_0)^2u''(.)\,f(x_1/r)\,g(x_2)\,dx_1dx_2\frac{d\alpha_1^*}{dr}\\
&+\int_{\underline{x}_1}^{\overline{x}_1}\int_{\underline{x}_2}^{\overline{x}_2}(x_1-x_0)(x_2-x_0)\,u''(.)\,f(x_1/r)\,g(x_2)\,dx_1dx_2\frac{d\alpha_2^*}{dr}\\
&+\int_{\underline{x}_1}^{\overline{x}_1}\int_{\underline{x}_2}^{\overline{x}_2}(x_1-x_0)\,u'(.)\,f_r(x_1/r)\,g(x_2)\,dx_1dx_2\\
=\;&0. \qquad (4)
\end{aligned}$$

The second term in the above equation can be rewritten as:

$$\int_{\underline{x}_1}^{\overline{x}_1}\int_{\underline{x}_2}^{\overline{x}_2}(x_1-x_0)(x_2-x_0)\frac{u''(.)}{u'(.)}u'(.)\,f(x_1/r)\,g(x_2)\,dx_1dx_2. \qquad (5)$$

By the assumption of constant relative risk aversion (CRRA) we have:

$$(x_2-x_0)\frac{u''(.)}{u'(.)}=\frac{c}{\alpha_2^*}-\frac{\alpha_1^*}{\alpha_2^*}(x_1-x_0)\frac{u''(.)}{u'(.)}. \qquad (6)$$

Substituting (6) in (5), we get, after some simplifications:

$$\begin{aligned}
&\frac{c}{\alpha_2^*}\int_{\underline{x}_2}^{\overline{x}_2}\int_{\underline{x}_1}^{\overline{x}_1}(x_1-x_0)\,u'(.)\,f(x_1/r)\,g(x_2)\,dx_1dx_2\\
&-\frac{\alpha_1^*}{\alpha_2^*}\int_{\underline{x}_2}^{\overline{x}_2}\int_{\underline{x}_1}^{\overline{x}_1}(x_1-x_0)^2\,u''(.)\,f(x_1/r)\,g(x_2)\,dx_1dx_2. \qquad (7)
\end{aligned}$$

The first term in (6) is nil by the first order condition associated to the choice of α_1.

The expression in (4) can now be written as:

$$\begin{aligned}
&\int_{\underline{x}_1}^{\overline{x}_1}\int_{\underline{x}_2}^{\overline{x}_2} (x_1 - x_0)^2 u''(\cdot) f(x_1/r) g(x_2) dx_1 dx_2 \left[\frac{d\alpha_1^*}{dr} - \frac{\alpha_1^*}{\alpha_2^*}\frac{d\alpha_2^*}{dr}\right] \\
&+ \int_{\underline{x}_1}^{\overline{x}_1}\int_{\underline{x}_2}^{\overline{x}_2} (x_1 - x_0) u'(\cdot) f_r(x_1/r) g(x_2) dx_1 dx_2 \\
=\;& 0. \qquad (8)
\end{aligned}$$

Since

$$\frac{d\left(\frac{\alpha_2^*}{\alpha_1^*}\right)}{dr} = \frac{\alpha_1^*\frac{d\alpha_2^*}{dr} - \alpha_2^*\frac{d\alpha_1^*}{dr}}{(\alpha_1^*)^2},$$

then, by (8)

$$\operatorname{Sign}\left(\frac{d\left(\frac{\alpha_2^*}{\alpha_1^*}\right)}{dr}\right) = -\operatorname{Sign}\left(\int_{\underline{x}_1}^{\overline{x}_1}\int_{\underline{x}_2}^{\overline{x}_2} (x_1 - x_0) u'(\cdot) f_r(x_1/r) g(x_2) dx_1 dx_2\right).$$

Now we prove that

$$\int_{\underline{x}_1}^{\overline{x}_1}\int_{\underline{x}_2}^{\overline{x}_2} (x_1 - x_0) u'(\cdot) f_r(x_1/r) g(x_2) dx_1 dx_2 \le 0$$

under MLRP.

In fact,

$$\begin{aligned}
&\int_{\underline{x}_1}^{\overline{x}_1}\int_{\underline{x}_2}^{\overline{x}_2} (x_1 - x_0) u'(\cdot) f_r(x_1/r) g(x_2) dx_1 dx_2 \\
=\;& \int_{\underline{x}_1}^{\overline{x}_1}\left[(x_1 - x_0)\int_{\underline{x}_2}^{\overline{x}_2} u'(\cdot) g(x_2) dx_2\right] f_r(x_1/r) dx_1 \\
=\;& \int_{\underline{x}_1}^{\overline{x}_1} K(x_1) f_r(x_1/r) dx_1, \qquad (9)
\end{aligned}$$

where

$$K(x_1) = (x_1 - x_0)\int_{\underline{x}_2}^{\overline{x}_2} u'(\cdot) g(x_2) dx_2.$$

Note that $K(x_1) \begin{cases} \leq 0 & \forall x_1 \leq x_0 \\ \geq 0 & \forall x_1 \geq x_0 . \end{cases}$

Let's define

$$k(u) = \frac{K(u)\, f(u/r)}{-\int_{x_0}^{\overline{x}_1} K(v)\, f(v/r)\, dv} \quad \text{for } u \in [\underline{x}_1, x_0].$$

By the first order condition (2) we have:

$$\int_{x_0}^{\overline{x}_1} K(x_1)\, f_r(x_1/r)\, dx_1 = -\int_{\underline{x}_1}^{x_0} K(x_1)\, f_r(x_1/r)\, dx_1 \geq 0;$$

which implies that

$$\int_{\underline{x}_1}^{x_0} k(u)\, du = 1 \text{ and } k(u) \geq 0 \text{ for } u \in [\underline{x}_1, x_0]. \tag{10}$$

Now using (10) we can write the last term in (8) as:

$$\begin{aligned}
&\int_{\underline{x}_1}^{x_0} k(u) \frac{f_r(u/r)}{f(u/r)} \left\{ -\int_{x_0}^{\overline{x}_1} K(v)\, f(v/r)\, dv \right\} du \\
&+ \int_{x_0}^{\overline{x}_1} K(v)\, f(v/r) \frac{f_r(v/r)}{f(v/r)} \left\{ \int_{\underline{x}_1}^{x_0} k(u)\, du \right\} dv \\
= &-\int_{\underline{x}_1}^{x_0} \int_{x_0}^{\overline{x}_1} k(u)\, K(v)\, f(v/r) \frac{f_r(u/r)}{f(u/r)} du dv \\
&+ \int_{\underline{x}_1}^{x_0} \int_{x_0}^{\overline{x}_1} k(u)\, K(v)\, f(v/r) \frac{f_r(v/r)}{f(v/r)} du dv \\
= &\int_{\underline{x}_1}^{x_0} \int_{x_0}^{\overline{x}_1} k(u)\, K(v)\, f(v/r) \left(\frac{f_r(v/r)}{f(v/r)} - \frac{f_r(u/r)}{f(u/r)} \right) du dv.
\end{aligned} \tag{11}$$

Since $k(u) \geq 0$; $K(v) \geq 0$ for $u \in [\underline{x}_1, x_0]$; $v \in [x_0, \overline{x}_1]$ and by MLRP we also have

$$\frac{f_r(v/r)}{f(v/r)} \leq \frac{f_r(u/r)}{f(u/r)} \text{ for } (u, v) \in [\underline{x}_1, x_0] \times [x_0, \overline{x}_1].$$

The term in (11) is negative. Consequently, we have:

$$\frac{d\left(\frac{\alpha_2^*}{\alpha_1^*}\right)}{dr} \geq 0.$$

3 Conclusion

In this note, we have shown that it is not appropriate to limit the adjustment of total wealth between the risky portfolio and the safe asset following a FSD shift in a return distribution, even when the two-fund separation theorem holds. The investor must first evaluate the effect of the shift on the relative proportions of the risky assets in the risky portfolio and then decide how to adjust his total investment between the safe asset and the adjusted risky portfolio. The same conclusions holds for mean preserving spreads (Dionne, Gagnon and Dachraoui, 1997). Another conclusion is that the separation of conditions on both utility functions and distribution functions does not hold to obtain intuitive variations in risky assets following a distribution shift. In other words, it is not possible to limit conditions either on $u(\cdot)$ or on $F(x/r)$ to obtain the desired results. This means that the separating conditions presented in the finance literature hold for the setting of the optimal portfolios but are not robust to the comparative statics following distributional shifts if we want to obtain intuitive results.

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4 Appendix

Example 1 Suppose that the utility function is quadratic or that the returns distribution is normal and the utility function is exponential. Define $\{F(x_1/x_2,\ldots,x_n;r)\}_r$ as a mean preserving spread, then conditions in (1) are verified and the composition of the two funds remains stable following a mean preserving spread.

Let us start with the quadratic utility function.

We consider the last $n-2$ first order conditions:

$$
\left\{
\begin{array}{l}
\sum_{1}^{n-1} \alpha_j^*(.) \int \cdots \int (x_2 - x_n)(x_j - x_n)\, dF(x_1/x_2,\ldots,x_n;r)\, dG(x_2,\ldots,x_n) \\
\quad + \int \cdots \int x_n (x_2 - x_n)\, dF(x_1/x_2,\ldots,x_n;r)\, dG(x_2,\ldots,x_n) = 0 \\
\quad\quad \vdots \\
\sum_{1}^{n-1} \alpha_j^*(.) \int \cdots \int (x_{n-1} - x_n)(x_j - x_n)\, dF(x_1/x_2,\ldots,x_n;r)\, dG(x_2,\ldots,x_n) \\
\quad + \int \cdots \int x_n (x_{n-1} - x_n)\, dF(x_1/x_2,\ldots,x_n;r)\, dG(x_2,\ldots,x_n) = 0.
\end{array}
\right.
\tag{12}
$$

Since the increase in risk is a mean preserving spread then one can verify, under the ceteris paribus assumption[1], that

$$\int \cdots \int (x_l - x_n)(x_1 - x_n)\, dF(x_1/x_2,\ldots,x_n;r)\, dG(x_2,\ldots,x_n)$$

is independent of r for all $l = 2,\ldots,n-1$.

The same result applies for the terms

$$\int \cdots \int x_n (x_j - x_n)\, dF(x_1/x_2,\ldots,x_n;r)\, dG(x_2,\ldots,x_n) \quad \text{for } j = 2,\ldots,n-1$$

[1] On the ceteris paribus assumption see Meyer and Ormiston [1994] and Dionne and Gollier [1996].

and

$$\int \cdots \int (x_l - x_n)(x_j - x_n)\, dF(x_1/x_2, \ldots, x_n, r)\, dG(x_2, \ldots, x_n) \quad \text{for } l, j = 2, \ldots n-1.$$

We can write the system in (12) as:

$$\begin{cases} \sum_{1}^{n-1} a_j^2 \alpha_j^*(.) + a_n^2 = 0 \\ \vdots \\ \sum_{1}^{n-1} a_j^{n-1} \alpha_j^*(.) + a_n^{n-1} = 0. \end{cases}$$

The last system has $n-1$ parameters and $n-2$ equations that yield one degree of freedom. The solution of the above system can be written as:

$$\begin{cases} \alpha_2^*(.) = a_2 \alpha_1^*(.) + b_2 \\ \vdots \\ \alpha_{n-1}^*(.) = a_{n-1} \alpha_1^*(.) + b_{n-1}. \end{cases}$$

The most important fact here is that $a_2, \ldots, a_n, b_2, \ldots, b_n$ are independent of r.

Notice that if $b_j > 0$, for $j = 2, \ldots, n-1$, then we can extend the result of Meyer and Ormiston [1994] to the case of n-assets.

As an example we consider the case where $n = 3$. We find that:

$$\alpha_2^* = -\frac{\sigma_{33}^2 + (m_1 - m_3)(m_2 - m_3)}{(m_2 - m_3)^2 + \sigma_{22}^2 + \sigma_{33}^3} \alpha_1^* + \frac{\sigma_{33}^2 - m_3(m_2 - m_3)}{(m_2 - m_3)^2 + \sigma_{22}^2 + \sigma_{33}^3}, \quad (13)$$

where

$$\begin{aligned} m_i &= E(\tilde{x}_i), \\ \sigma_{ii}^2 &= var(\tilde{x}_i). \end{aligned}$$

As we can see from (13), the second term on the right hand side is negative for a range of the parameters m_2, m_3 and σ_{33}^2. As a result, even if the problem with three assets can be reduced to a problem with only two assets, we need to restrict the support of the two assets to be always positive if one wants to extend directly the result of Meyer and Ormiston [1994].

When the utility function is exponential and the returns distribution is normal, we use the Stein's lemma to write the last $n - 2$ first order conditions as:

$$
\begin{aligned}
& \mathrm{cov}\left(\alpha_1^*(x_1 - x_n) + \dots + \alpha_{n-1}^*(x_{n-1} - x_n) + x_n, x_j - x_n\right) \\
= \; & -\frac{E\left(u'\right)}{E\left(u''\right)} E\left(x_j - x_n\right), \text{ for } j = 2, \dots, n-1. \qquad (14)
\end{aligned}
$$

Since u is exponential then $-\frac{E\left(u'\right)}{E\left(u''\right)}$ is a constant and hence independent of r, and, with the same argument as in the previous example, the term on the left hand side of (14) is independent of r. The rest of the proof is as for the quadratic utility function.