Some Remarks About the Probability Weighting Function

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Résumé

Cet article analyse les implications des tests les plus fondamentaux pour les loteries sur la courbe de transformation des probabilités w(p). Nous montrons que pour les choix standards de loteries, la fonction w(p) a une forme en S (d'abord concave puis convexe) mais n'est pas régressive. Pour les tests où l'on doit donner un prix à une loterie la fonction w(p) a encore une forme en S mais est régressive. Dans la dernière section, nous proposons une solution qui permet d'accomoder les restrictions imposées par les tests considérés dans cette recherche.

Mots clés : Courbe de transformation des probabilités w(p), choix standards de loteries, fonction w(p), fonction régressive, tests, prix de loterie.

Classification JEL : D80.

Abstract

This paper analyses the implications of basic tests for lotteries on the probability weighting function w(p). We first show that the three standard tests for lottery choices imply that the w(p) function has a S-shape (first concave then convex) but is not regressive. For the pricing of lotteries the function has still a S-shape but is regressive. In the last section we propose a solution that accomodates the restrictions imposed by the choice tests and the pricing tests.

Keywords : Probability weighting function w(p), standard tests for lottery choices, w(p) function, regressive function, tests, pricing of lotteries.

JEL Classification : D80.

1. INTRODUCTION

In a recent article, Prelec (1998) presented axioms for several probability weighting functions which are regressive (intersecting the diagonal from above) and S-shaped (first concave, then convex). The S-shape curves are also supported by empirical evidence in many situations (Wu and Gonzalez, 1996,1998; Camerer and Ho, 1994; and Tverski and Fox, 1994). These functions can be obtained from subadditivity (Tverski and Wakker, 1995), or from subproportionality (Prelec, 1998).

In this note we consider two-point lotteries (x,p) where the decision maker can win the price $x > 0$ with probability $p \in]0,1[$ and nothing otherwise. The evaluation function of this lottery is w(p) u(x) where u(x) is strictly increasing. This simple evaluation function for the lottery (x,p) is used in several models including cumulative prospect theory (Tverski and Kahneman, 1992) which is axiomatized in Wakker and Tverski (1993). We analyse how the results of different tests for lottery choices and pricing affect the w(p) function.

For this two-point lottery there are five different basic tests. For the choice between two lotteries (x_1,p_1) and (x_2,p_2) there exist three possible tests: The first one is where the two probabilities of winning are high and this is noted (H,H); the second one (L,L) is where the two probabilities of winning are low; and, the third one (H,L) is where one probability is high and the other one is low. For the pricing of a lottery (x,p) the decision maker must give a minimum selling price for the lottery; this price corresponds to the certainty equivalent. We consider two possible tests: the test where the probability of winning is high and the test where the probability of winning is low. These two tests combined with the (H,L) test define the preference reversal paradox.

In this note, as we focus on the probability weighting function, we assume that w(p) is a necessary condition to explain the results of the five tests above. As in Prelec (1998) and Tverski and Wakker (1995), we assume that the function u(x) is strictly increasing, so the explanations of the results of the five tests are attributed to the w(p) function alone. Tverski and Kahneman (1992) obtained an S-shape curve for c/x as a function of p, where c is the certainty equivalent of the lottery (x,p). Even if u(x) concave with w(p) linear can explain the convex part of the curve or u(x) convex with w(p) linear can explain the concave part they assumed that the function w(p) is a necessary condition to explain both parts of the curve c/x. The analysis of the data confirms their hypothesis.

In this paper we first show that the w(p) function is S-shaped but does not have to be regressive to solve the first two tests for lottery choice. We then show that there exists a function w(p) that can accomodate the three tests of choice and this function cannot be regressive. We also specify the shape of the w(p) function for the pricing of a lottery. We obtain that the function is S-shaped and regressive. In the last section we propose a more general probability weighting function that accomodates the restrictions imposed by the five basic tests. All the proofs are in the Appendix.

2. IMPLICATIONS OF THE TESTS

2.1 Tests (H,H) and (L,L)

An example of choice (L,L) in Luce et al. (1992) is (.17, 3) vs (.09, 5.4) where most subjects choose the second lottery. An example of choice (H,H) in Leland (1994) is (.71, 10.88) vs (.79, 9.67) where 84% of the subjects choose the second lottery. These two tests can also be founded simultaneously in the common-ratio paradox where, however, the ratio of probabilities is the same for each test. An example of this paradox in Kahneman and Tverski (1979) is given below:

Problem 7: Choose between

A_1 (.1, 0; .9, 3000) and A_2 (.55, 0;.45, 6000).

Problem 8: Choose between

B_1 (.998, 0;.002, 3000) and B_2 (.999, 0;.001, 6000).

As documented by the authors, 86 per cent of the subjects choose A_1 in Problem 7, and 73 per cent of the subjects chose B_2 in Problem 8, which contradicts the expected utility paradigm. In Problem 7, the probabilities of winning are high (.90 and .45), so the individuals choose the prospect where winning is more probable. In Problem 8, the probabilities of winning are very low and most people choose the prospect that offers the larger gain (see MacCrimmon and Larsson, 1979, for similar results). As we assume that a transformation of p is a necessary condition to explain the Luce et al (1992) and Leland (1994) tests, we have the following condition for the two lotteries $(x_1,p+\Delta)$ and (x_2,p) for the test (H,H):

Condition 1: $(p+\Delta)\, u(x_1) - p\, u(x_2) = 0$

$$w(p+\Delta)\, u(x_1) - w(p)\, u(x_2) > 0 \quad \forall\, \Delta>0, \forall p \,/\, 1>p+\Delta>p>a_1.$$

And for the test (L,L):

Condition 2: $(p+\Delta)\, u(x_1) - p\, u(x_2) = 0$

$$w(p+\Delta)\, u(x_1) - w(p)\, u(x_2) < 0 \quad \forall\, \Delta>0, \forall p \,/\, 0<p<p+\Delta<a_1.$$

The next theorem identifies the different possible forms of a probability weighting function that will satisfy Conditions 1 and 2 if the function w(p) is sufficiently regular (the function has at most one inflection point).

Theorem 1: *Let w(p), wℕ(p) and w0(p) be continuous and assume that there exists at most one p 0 [0,1] such that w0(p) = 0. Let also w(0) = 0, w(1)=1, wℕ(p)>0. Conditions 1 and 2 are satisfied if and only if there exists a* a_0 *such that w0(p) < 0 on]0,*a_0*[and w0(p) > 0 on]*a_0*,1[and there exists a point* b_1 *such that w(p) < p for all p 0]*b_1*,1[.*

The two forms that correspond to the theorem are presented in Figure 1:

(Insert Figure 1 about here)

We observe from Figure 1 that the function w(p) does not have to intersect the diagonal on]0,1[to solve the first two tests. However, there exists a b_1 such that w(p) < p for all p ∈]b_1,1[. In other words the function w(p) cannot be above the diagonal for all p. The Type 1 function is regressive (there exist $p^*_1 < p^*_2 < p^*_3$ such that $w(p^*_1) > p^*_1$, $w(p^*_2) = p^*_2$, $w(p^*_3) < p^*_3$ where p^*_2 is the unique fixed point on]0,1[) but the Type 2 function is not regressive and there exists a b_2 such that w(p) < p for all p ∈]0,b_2[. The existence of such a b_2 combined with an S-shape curve implies that w(p) < p for all p ∈]0,1[. For any function, such as that proposed by Tverski and Kahneman (1992), $w(p) = p^\delta/(p^\delta + (1-p)^\delta)^{1/\delta}$, concavity followed by convexity implies $\delta < 1$. But $\delta < 1$ also implies the existence of a fixed point. Consequently, the test on data for (H,H) and (L,L) lotteries with this function cannot conclude on the existence of a fixed point (See also Prelec, 1998, on this issue). A function that may be a candidate to test the existence of a fixed point is $ap^3 + bp^2 + cp$. When a = -b > 0 and c=1 this curve is like the Type 2 function in Figure 1. However, with other values of the parameters, it can fit the Type 1 curve. One implicit conclusion from the above analysis is that the w(p) function does not have to be regressive in order to solve the common-ratio paradox.

2.2 Test (H,L) and Pricing

The preference reversal paradox, as tested by Tverski et al (1990), takes into account the two tests of pricing ($p<a_3$, $p>a_3$) and the choice (H,L). For example, they tested the lotteries (.97,4) and (.31,16), where the two lotteries have comparable expected value. When a choice between the two lotteries is offered, 83% of the subjects choose the lottery (.97,4). However when asked to state their lower selling price 74% of the subjects stated a higher price for (.31,16) than for (.97,4). The choice (.97,4) vs (.31,16) implies the following condition:

Condition 3: $(p+\Delta)\, u(x_1) - p\, u(x_2) = 0$

$w(p+\Delta)\, u(x_1) - w(p)\, u(x_2) > 0 \;\forall\Delta>0,\ \forall p \,/\, 0<p<a_2<p+\Delta<1$

We now show that the function w(p) cannot intersect the diagonal when we add this type of lottery choice used in preference reversal to the first two tests. In other words, the function Type 1 in Figure 1 is no longer possible when the three choices are analysed together.

Theorem 2: *Let $w(p)$, $w'(p)$ and $w''(p)$ be continuous and assume that there exists at most one $p \in [0,1]$ such that $w''(p) = 0$. Let also $w(0) = 0$, $w(1)=1$, $w'(p)>0$, and $w(p)$ satisfying Conditions 1,2. Thus Condition 3 is satisfied if and only if there exists a b_2 such that $w(p) < p$ for all $p \in]0,b_2[$.*

The next corollary shows that if Conditions 1, 2 and 3 are satisfied, then a fixed point does not exist on the interval]0,1[.

Corollary 1: *Let $w(p)$, $w'(p)$ and $w''(p)$ be continuous and assume that there exists at most one $p \in [0,1]$ such that $w''(p) = 0$. Let also $w(0) = 0$, $w(1)=1$, $w'(p)>0$, and $w(p)$ satisfy Conditions 1 and 2. Thus there exists a b_2 such that $w(p) < p$ for all $p \in]0,b_2[$ if and only if $w(p) < p$ for all $p \in]0,1[$.*

Consequently, only functions of Type 2 in Figure 1 are compatible with the three lottery choices. We still have an S-shape curve, but there is no fixed point. However, as is well established, the preference reversal paradox alone is enough to imply the absence of a w(p) function. For the example above we have for the pricing:

$$w(.97)\,u(4) - w(.31)\,u(16) < 0$$

and for the choice:

$$w(.97)\ u(4) - \ w(.31)\ u(16) > 0.$$

These two conditions imply:

$$w(.97)/w(.31) > w(.97)/w(.31)$$

and there does not exist a probability weighting function w(p) that can accomodate this paradox. Nevertheless we will study the implications of lottery pricing on the w(p) function.

Tverski and Kahneman (1992) have shown that the certainty equivalent c of a lottery (x,p) is such that c/x is greater than p for low p and lower than p for high p. If w(p) is necessary to explain this fact then we find that w(p) u(x) >(<) p u(x) for p <(>) a_3. We now show that the two tests of pricing imply an S-shape curve that is regressive.

Theorem 3: *Let w(p), wℕ(p) and w0(p) be continuous and assume that there exists at most one p 0 [0,1] such that w0(p) = 0. Let also w(0) = 0, w(1)=1, wℕ(p)>0. Then w(p) u(x) >(<) p u(x) for p <(>) a_3 if and only if w(p) is S-shaped and is regressive (Type 1 in Figure 1).*

Consequently, the w(p) function is regressive for pricing contrary to lottery choices. We can note that, for pricing, the discontinuity of w(p) at p = 0 is possible. Moreover, as pointed out by Prelec (1998), the discontinuity gives a qualitative character to the transition from impossibility to possibility. However, for the choice in preference reversal this discontinuity cannot accomodate Condition 3.

3. DISCUSSION

We have shown that the w(p) function cannot be regressive for the three choice tests but must be regressive for the two pricing tests. Consequently, there does not exist a w(p) function that can solve simultaneously the five basic tests. One explanation of this result may be that the decision-maker uses a decision process quite different than w(p) u(x). Another explanation is to consider that the w(p) function with one argument is too restrictive. One extension would be to study a function $w(p_1;p_2)$ that evaluates p_1 while taking into account the probability p_2 . For example in comparing two lotteries (.31, 16) vs (.97, 4) the evaluation function would become w(.97;.31) u(4) - w(.31;.97) u(16).

Consider now the pricing of lotteries. Suppose that the individual has to price the lottery (x_1,p_1). The question is: Which probability p_2 will he use to evaluate $w(p_1;p_2)$? Two evident candidates are the boundaries p=1 and p=0. For example, for a lottery with a low winning probability (.31, 16), the pricing can be done by using the nearest boundary p = 0, so that the evaluation function is equal to w(.31;0) u(16); while, for a high winning probability (.97, 4), the probability can be evaluated by using the nearest boundary p = 1 so that w(.97;1) u(4).

With such a $w(p_1;p_2)$ function the choice between lotteries (.31, 16) vs (.97, 4) for preference reversal implies:

$$w(.97;.31)/w(.31;.97) > u(16)/u(4)$$

and for the pricing we have:

$$w(.97;1)/w(.31;0) < u(16)/u(4)$$

so that we obtain:

$$w(.97;.31)/w(.97;1) > w(.31;.97)/w(.31;0).$$

This condition solves the preference reversal paradox. A possible extension is to test whether, for the domain]0,1[×]0,1[, we have a p_l such that $w(p_l;p_j) = k$ for all p_j. This function is possible with Theorem 2.

There are several ideas in the literature that are able to justify the conditions imposed by the basic tests. The curve for the pricing is the same as in Wakker and Tverski (1995) and may be explained by subadditivity. Slovic and Lichtenstein (1983) consider that choices among gambles appear to be influenced primarily by probabilities in the preference reversal paradox which is equivalent to Condition 3. An explanation of the first two tests which is equivalent to Conditions 1 and 2 is Rule 6 in MacCrimmon and Larsson (1979).

A more fundamental extension is to build up a unified model that will take into account these three ideas and explain why the decision maker changes his way of judging probabilities.

APPENDIX

The proofs of Lemmas and Theorems use the Mean Value Theorem (MVT) and the Intermediate Value Theorem (IVT). The proof of MVT can be found in any calculus text and that of IVT can be found in Munkres (1975).

Proof of Theorem 1: The next lemma shows that if the function w(p) is sufficiently regular (the function has at most one inflection point), Conditions 1 and 2 defined with two points p and p+Δ are implied by conditions associated to one point which is more easily tractable.

Lemma 1: *Let $w(p)$, $w\mathbb{N}(p)$ and $w0(p)$ be continuous and assume that there exists at most one p 0 [0,1] such that $w0(p) = 0$. Let also $w(0) = 0$, $w(1)=1$, $w\mathbb{N}(p)>0$. If $w\mathbb{N}(p) >(<) w(p)/p$ for all p such that $1>p>a_1$ $(0<p<a_1)$ then Condition 1 (2) is satisfied.*

Proof of Lemma 1: For Condition 1 we have to prove that if there exists a $p_1 > a_1$ and a $\Delta_1>0$ such that $1>p_1+\Delta_1>p_1>a_1$ and $w(p_1+\Delta_1)/w(p_1) \leq (p_1+\Delta_1)/p_1$, then there exists a p such that $w'(p) \leq w(p)/p$. If $w'(p_1) \leq w(p_1)/p_1$, we have this p. If $w'(p_1) > w(p_1)/p_1$ there exists an interval $[p_1,p_2]$ where p_2 is the first point such that $w(p_2) = w(p_1) + (p_2-p_1)(w(p_1+\Delta_1) - w(p_1))/\Delta_1$. Such p_2 exists because $p_1 + \Delta_1$ has this property. For all $p \in]p_1,p_2[$, w(p) is above the straight line through (p_1,

$w(p_1))$ and $(p_1+\Delta_1, w(p_1+\Delta_1))$. First $w(p_1) + \Delta_1 w(p_1)/p_1 \geq w(p_1+\Delta_1)$ implies $w(p_1)/p_1 \geq (w(p_1+\Delta_1) - w(p_1))/\Delta_1 = m$. Thus $w'(p_1) > m$ and since $w'(p)$ is continuous there exists an interval $[p_1,p_1+\epsilon]$such that $w'(p) > m$ for all p and then $w(p) > w(p_1) + m(p-p_1)$ on $]p_1,p_1+\epsilon]$. If not, MVT implies that there exists a p such that $w'(p) = (w(p) - w(p_1))/(p-p_1) \leq m$ which is in contradiction with $w'(p) > m$ for all $p \in [p_1,p_1+\epsilon]$. As there exists a $w(p) > w(p_1) + m\ (p-p_1)$ and as p_2 is the first point where $w(p) = w(p_1) + m\ (p-p_1)$, then all points on $]p_1,p_2[$ are above the straight line. If not, then IVT implies that there exists another $p < p_2$ such that $w(p) = w(p_1) + m\ (p-p_1)$. Moreover MVT implies that there exists a $p_3 \in]p_1, p_2[$ such that $w'(p_3) = m$ and, as $w(p_3) > w(p_1) + m(p_3-p_1)$, we have $w(p_3) - mp_3 > w(p_1) - mp_1 \geq 0$ and then $m < w(p_3)/p_3$. Thus $w'(p_3) < w(p_3)/p_3$. The proof for Condition 2 is similar.

Lemma 2 will be used frequently in the next proofs.

Lemma 2: *Let w(p), wN(p) and w0(p) be continuous. If there exists a p such that w(p) >(<)) 1m + w(p-) 1) where m = (w(p+) 2) - w(p-) 1))/() 1+) 2), and) 1,) 2 > 0, then there exists a p1 0]p-) 1,p+) 2[such that w0(p1) <(>) 0.*

Proof of Lemma 2: MVT implies that there exists a point $p_2 \in]p-\Delta_1,p[$ such that

$$w'(p_2) = (w(p) - w(p-\Delta_1))/\Delta_1 > m.$$

MVT implies that there exists a point $p_3 \in\ ]p,p+\Delta_2[$ such that

$$w'(p_3) = (w(p+\Delta_2)-w(p))/\Delta_2 < m.$$

MVT implies that there exists a $p_1 \in\ ]p_2,p_3[$ such that $w''(p_1) = (w'(p_3)-w'(p_2))/(p_3-p_2) < 0$.

The proof for the case (>) is similar.

□

Lemma 3 states one condition that is equivalent to a w(p) function with a S-shape (first concave, then convex) when there exists a b_1 such that $w(p) < p$ for all $p \in\ ]b_1,1[$.

Lemma 3: *Let w(p), w𝕅(p) and w0(p) be continuous and assume that there exists at most one p 0 [0,1] such that w0(p) = 0. Let also w(0) = 0, w(1)=1, w𝕅(p) < 0 and that there exists a* b_1 *such that w(p) < p for all p 0* $]b_1,1[$ *. Then there exists a* a_4 *such that w0(p) < 0 on* $[0,a_4[$ *if and only if there exists a* a_0 *such that w0(p) < 0 on* $[0,a_0[$ *and w0(p) > 0 on* $]a_0,1]$*.*

Proof of Lemma 3: (⇒) As $w(0) = 0$ and $w(1) = 1$ and as there exists a p_0 such that $w(p_0) < p_0$, Lemma 2 implies that there exists a p_1 such that $w''(p_1) > 0$. But as there exists a a_4 such that $w''(p) < 0$ for all $p \in [0,a_4[$, IVT implies that there exists a point $a_0 \in\ ]0,p_1[$ where $w''(a_0) = 0$. Thus $w''(p) > 0$ on $]a_0,1[$. If not, there exists a point p_2 such that $w''(p_2) \leq 0$. $w''(p_2) = 0$ is impossible since $w(a_0) = 0$

and there exists only one point of inflection. If $w''(p_2) < 0$, as we have $w''(p_1) > 0$, IVT implies that there exists a point $p_3 > a_0$ such that $w''(p_3) = 0$ which is also impossible. Thus $w''(p) > 0$ on $]a_0,1[$.

$w''(p) < 0$ for all $p \in [0,a_0[$. If not, there exists a $p \neq a_0$ such that $w''(p) = 0$ which is impossible since $w''(a_0) = 0$. If there exists a $p_4 \in [0,a_0[$ such that $w''(p_4) > 0$, and as $w''(p) < 0$ on $[0,a_4[$, IVT implies that there exists a $p \in]a_4,a_0[$ such that $w''(p) = 0$ which is a contradiction.

($\Leftarrow$) If $w''(p) < 0$ on $]0,a_0[$ then there exists a $a_4 \leq a_0$ such that $w''(p) < 0$ for all $p \in]0,a_4[$.

□

We can now prove Theorem 1.

($\Rightarrow$) If there does not exist a point b_1 such that $w(p) < p$ for all $p \in]b_1,1[$, then there exists a p such that $w(p) \geq p$ for all intervals $]b_1,1[$. If there does not exist a p such that $w(p) > p$, then there exists a p_1 such that $w(p_1) = p_1$ and, for the interval $]p_1+\epsilon,1[$, there exists a p_2 such that $w(p_2) = p_2$. Thus $w(p_2)/p_2 = w(p_1)/p_1$ and Condition 1 is not satisfied. Thus there exists a point p_0 such that $w(p_0) > p_0 + \delta$. The continuity of $w(p)$ and p and $w(1) = 1$ imply that for δ there exists a ϵ such that $|(w(1)-1) - (w(p)-p)| < \delta$ when $1-p < \epsilon$. Thus, as $p > p_0$ we obtain $w(p_0)/p_0 > w(p)/p$ for all $p \in]1-\epsilon,1[$ which is in contradiction with Condition 1.

By lemma 3, the negation of a S-shape curve is equivalent to the fact that for all intervals $[0,a_4[$, there exists a $p_1 \in [0,a_4[$ such that $w''(p_1) \geq 0$. If $w''(0) = 0$ then $w''(p) > 0$ for all $p > 0$ or $w''(p) < 0$ for all $p > 0$. If not, there exist p_1 and p_2 such that $w''(p_1) > 0$ and $w''(p_2) < 0$ and then IVT implies that there exist a point $p_3 \neq 0$ such that $w''(p_3) = 0$ which is impossible. If $w''(p) > 0$ then for all $p > 0$, $w(p) - pw'(p) < w(0) = 0$ which is in contradiction with Condition 2 by Lemma 1. If $w''(p) < 0$ for all $p > 0$ then $w(p) - pw'(p) > 0$ and consequently Condition 1 is not satisfied.

Thus if for all a_4 there exists a point $p_1 \in]0,a_4[$ such that $w''(p_1) \geq 0$ then $w''(0) > 0$. If not, $w''(0) < 0$ along with continuity imply that there exists an interval $[0,a_4[$ where $w''(p) < 0$. Thus $w''(0) > 0$ along with continuity imply that there exists an interval $[0,a_4[$ where $w''(p) > 0$ and we have $w(p) - w'(p)\, p < 0$. Then $w'(p) > w(p)/p$ and Lemma 1 implies that Condition 2 is not satisfied.

($\Leftarrow$) $w''(p) < 0$ on $]0,a_0[$ implies $0 = w(0) < w(p) - pw'(p)$ which is Condition 2 by Lemma 1. Thus Condition 2 is satisfied on $]0,a_0[$. Now we have to prove that there exists a point a_1 such that $w'(a_1) = w(a_1)/a_1$. As there exists a p such that $w(p) < p$ on $]b_1,1[$ and $w(1) = 1$, then MVT implies that there exists a p_1 such that

$w'(p_1) = (1-w(p))/(1-p) > 1$. Furthemore, we have $1 > w(p_1)/p_1$. As the functions $p, w(p)$ and $w'(p)$ are continuous and $w(p) - pw'(p) > 0$ on $]0,a_0[$ and $w(p_1) - p_1 w'(p_1) < 0$, IVT implies that there exists a a_1 such that $w'(a_1) = w(a_1)/a_1$ and $a_1 \geq a_0$.

Now we prove that all points on $]a_1,1[$ satisfy Condition 1. Let $ka_1 = w(a_1)$ then $w'(a_1) = k$. Let $k_2(a_1+\Delta) = w(a_1+\Delta)$ then $k_2(a_1+\Delta) = w(a_1+\Delta) > w(a_1) + w'(a_1)\,\Delta = k(a_1 + \Delta)$. Thus $k_2 > k$. As $w''(p) > 0$ on $]a_1,1[$ then there exists a $k_1 > k$ such that $w(a_1+\Delta) = w(a_1) + \Delta k_1$. Thus $w(a_1+\Delta) = k_2(a_1+\Delta) = ka_1 + \Delta k_1$, and then $0 < a_1(k_2-k) = \Delta(k_1-k_2)$ and $k_1 > k_2$. As MVT implies that there exists a p such that $a_1 + \Delta > p > a_1$, $w'(p) = k_1$. As $w''(p) > 0$ we also have $w'(a_1+\Delta) > k_1$. If not, MVT implies that there exists a $p > a_0$ such that $w''(p) \leq 0$. Thus $w'(a_1+\Delta) > k_1 > k_2 = w(a_1+\Delta)/(a_1+\Delta)$ which is Condition 1. The proof for the case $a_0 \leq p < a_1$ where we have Condition 2 is similar. This completes the proof of Theorem 1. □

Proof of Theorem 2: ($\Rightarrow$) If there exists a p such that $w(p) \geq p$ for all intervals $]0,b_2[$, then there exists a p such that $w(p) > p$. If not, let p_1 be a point such that $w(p_1) \geq p_1$. If $w(p_1) > p_1$ we have this point. If not, $w(p_1) = p_1$ and for the interval $]0,p_1-\epsilon]$, there exists a p_2 such that $w(p_2) \geq p_2$. If $w(p_2) = p_2$, then $w(p_2)/p_2 = w(p_1)/p_1$ and Condition 2 is not satisfied. Thus there exists a p such that $w(p) > p$. Let p_3 be this point then $w(p_3) - p_3 = \delta > 0$. Thus continuity of both $w(p)$ and p and the fact that $w(1) = 1$ imply that for δ there exists a ϵ such that $|(w(1)-1) -$

$(w(p)\text{-}p)| < \delta$ when $1\text{-}p < \epsilon$. Thus as $p > p_3$ we obtain $w(p_3)/p_3 > w(p)/p$ for all $p \in [1\text{-}\epsilon,1[$ which is in contradiction with Condition 3.

($\Leftarrow$) Condition 3 is equivalent to the fact that there exists a a_2 such that $w(p)/p > m$ for all $p > a_2$ with equality at a_2 and $w(p)/p < m$ for all $p < a_2$. Let m be such that $w'(p) < m$ for $p \in]0,b_2[$ when $b_2 < a_0$, then $m \leq 1$. If not, then both $w''(p) < 0$ and $w(0) = 0$ imply that $w(p) - w'(p)p > 0$. Consequently $w(p) > p$ which is in contradiction with the definition of a Type 2 curve. Moreover m is maximal on $]0,a_0]$. If not, MVT contradicts $w''(p) < 0$ on $]0, a_0[$. We now prove that a Type 2 curve implies Condition 3.

As $w(p) < mp$ when $p \in]0,a_0]$, then $ma_2 = w(a_2)$ implies that $w''(a_2) > 0$. a_2 exists since $w(a_0) < ma_0$, $w(1) > m$ and continuity of $w(p)$ imply that there exists a function $h(p)$ such that $h(1) = w(1)\text{-}m > 0$ and $h(a_0) = w(a_0) \text{-}ma_0 < 0$. Thus IVT implies that there exists a $a_2 \in]a_0,1[$ such that $h(a_2) = 0$ and then $w(a_2) = ma_2$.

$w'(a_2) > m$ since $w(a_0) < ma_0$ and $w(a_2) = ma_2$. MVT implies that there exists a $p_3 < a_2$ such that $w'(p_3) = (w(a_2) - w(a_0))/(a_2\text{-}a_0) > m$ and $w''(p) > 0$ implies $w'(a_2) > m$. If not, as $w'(p_3) > m$, MVT implies that there exists a $p_4 \in]p_3,a_2[$ such that $w''(p_4) < 0$ which is impossible.

For all points $p > a_2$ we have:

$$w(p) > w'(a_2)\,(p-a_2) + w(a_2)$$

$$w(p) > m(p-a_2) + ma_2$$

$$w(p)/p > m$$

For $]a_0,a_2]$ we have $w(p)/p < m$. If not, there exists a p_1 such that $h(p_1) = w(p_1) - mp_1 \geq 0$, $h(a_2) = 0$ and $h(a_0) < 0$. Lemma 2 implies that there exists a point where $w''(p) < 0$ which is false.

□

Proof of Corollary 1: (⇒) If Conditions 1 and 2 are satisfied then there exists a b_1 such that $w(p) < p$ for all $p \in \,]b_1, 1[$. Assume that there exists a b_2 such that $w(p) < p$ for all $p \in \,]0,b_2[$. Let $p_1 \in [b_2,b_1]$ be such that $w(p_1) \geq p_1$. As $w(p_1) \geq p_1$, $w(0) = 0$ and $w(p) < p$ on $]0,b_2[$, Lemma 2 implies that there exists a p_3 such that $w''(p_3) > 0$. But $w''(p) \leq 0$ for all $p \leq a_0$ and MVT implies that $a_0 < p_3$ and then $w''(p) > 0$ for $p \in [p_3, 1]$. If not, IVT implies that there exists a $p \neq a_0$ such that $w''(p) = 0$. As $w(p) < p$ for $p \in \,]0, b_2[$ and $w(p_1) \geq p_1$, MVT implies that there exists a p_4 such that $w'(p_4) > 1$ and then $w'(p) > 1$ for all $p \in [p_1, 1]$. If not there exists a p such that $w''(p) < 0$ by MVT which is a contradiction. Thus $w(1) > w(p_1) + (1-p_1) \geq 1$ which contradicts $w(1) = 1$.

(⇐) Evident.

Proof of Theorem 3: (⇒) $w(p) > p$ for all p on $]0,a_3[$ and $w(p) < p$ for all p on $]a_3,1[$ imply that both p^*_1 and p^*_3 exist. If $w(p) > p$ for all p on $]0,a_3[$ and $w(p) < p$ for all p on $]a_3,1[$, then IVT implies that there exists a fixed point. The sole candidate is a_3 and $a_3 = p^*_2$. As $w(p) > p$, $w(0) = 0$, and $w(p^*_2) = p^*_2$, Lemma 2 implies that there exists a point p_1 such that $w''(p_1) < 0$. As $w(p) < p$, $w(1) = 1$, and $w(p^*_2) = p^*_2$, Lemma 2 implies that there exists a point p_2 such that $w''(p_2) > 0$. Thus IVT implies that there exists a a_0 such that $w''(a_0) = 0$. For all p <(>) a_0, $w''(p)$ <(>) 0. If not, IVT implies that there exists another point such that $w''(p) = 0$.

(⇐) As we have a unique fixed point p^*_2, then $a_3 = p^*_2$. $w(p) > p$ for all $p \in]0,a_3[$. If not we have either a p such that $w(p) = p$ which implies another fixed point or a p such that $w(p) < p$. As $w(p^*_1) > p^*_1$, IVT implies that there exists another fixed point which is impossible. The proof of the case $w(p) < p$ on $]a_3,1[$ is similar.

□

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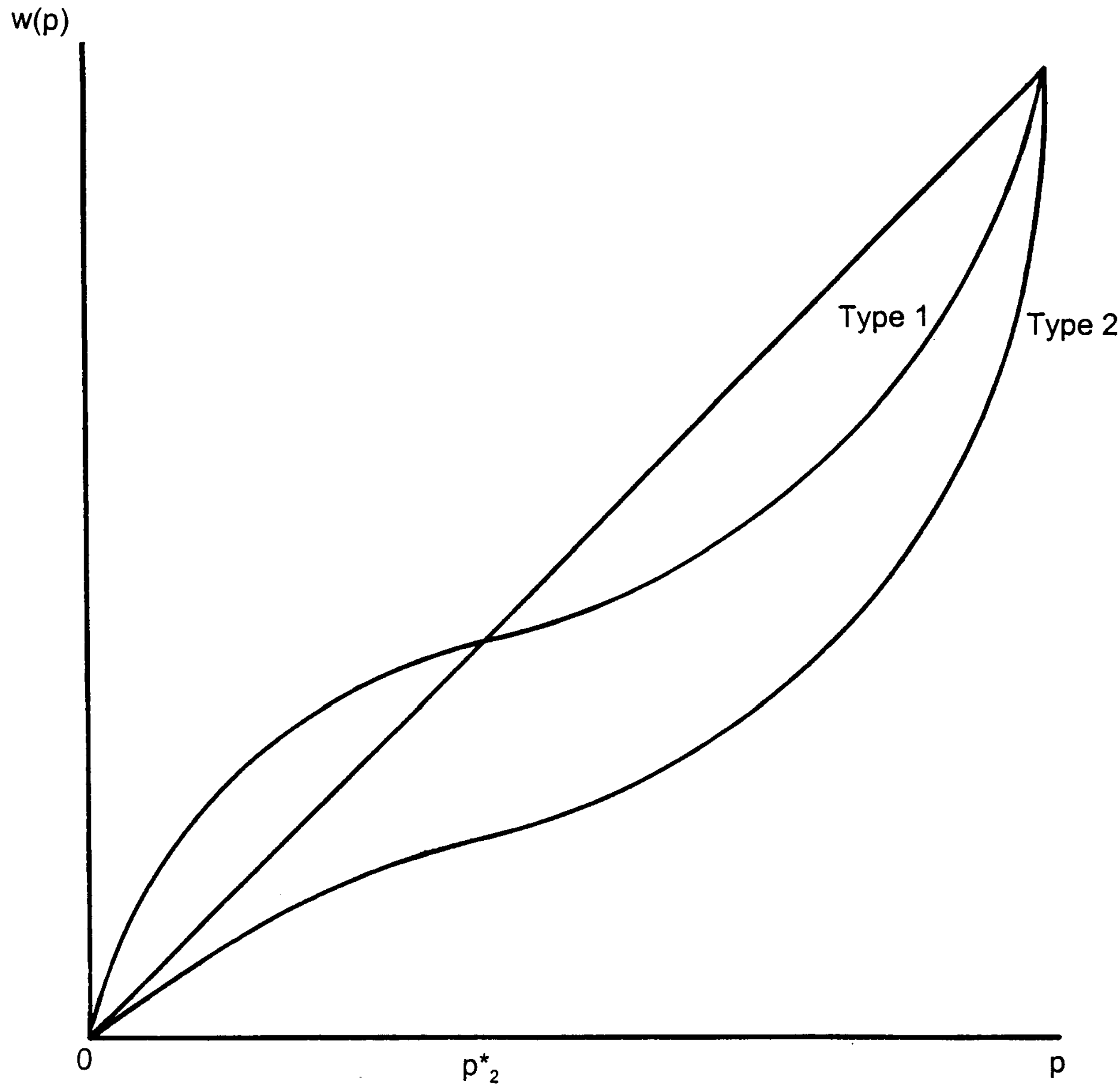

FIGURE 1.— Probability weighting function.