Coherent Diversification Measures in Portfolio Theory: An Axiomatic Foundation *

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Abstract

This paper provides an axiomatic foundation of the measurement of diversification in a one-period portfolio theory under the assumption that the investor has complete information about the joint distribution of asset returns. Four categories of portfolio diversification measures can be distinguished: the *law of large numbers* diversification measures, the *correlation* diversification measures, the *market portfolio* diversification measures and the *risk contribution* diversification measures. We offer the first step towards a rigorous theory of *correlation* diversification measures. We propose a set of nine desirable axioms for this class of diversification measures, and name the measures satisfying these axioms *coherent diversification measures* that we distinguish from the notion of coherent risk measures. We provide the decision-theoretic foundations of our axioms by studying their compatibility with investors' preference for diversification in two important decision theories under risk: the expected utility theory and Yaari's dual theory. We explore whether useful methods of measuring portfolio diversification satisfy our axioms. We also investigate whether or not our axioms have forms of representation.

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Keywords : Portfolio Theory, Portfolio Diversification, Preference for Diversification, Correlation Diversification, Expected Utility Theory, Dual Theory.

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1 Introduction

Diversification is one of the major components of decision making in conditions of risk and uncertainty, especially in portfolio theory (see Markowitz, 1952; Ross, 1976; Sharpe, 1964). It consists of investing in various assets. Its objective is to reduce risk, particularly the likelihood and severity of portfolio loss, through *multilateral insurance* in which each asset is insured by the other assets. Despite criticism after the 2007-2009 financial crisis (see Holton, 2009), it is still an important risk management tool for many institutions and regulators (see Basel Committee on Banking Supervision, 2010, 2013; Committee of European Banking Supervisors, 2010; Committee of European Insurance and Occupational Pensions Supervisors, 2010a,b; Committee on Risk Management and Capital Requirements, 2016; European Insurance and Occupational Pensions Authority, 2014; Ilmanen and Kizer, 2012; Laas and Siegel, 2017; Markowitz et al., 2009; Sandstrom, 2011). Its measurement and management, outside the standard risk measurement frameworks (e.g. the theory of monetary risk measures of Artzner et al. (1999) and Föllmer and Weber (2015)), remains of fundamental importance in finance and insurance economics.

1.1 Existing Diversification Measures

Following Markowitz (1952)'s pioneering work on the mathematical formulation of diversification in portfolio theory, several measures of portfolio diversification have been proposed. According to Koumou (2018), there are four categories of diversification measures: the *law of large numbers* diversification measures, the *correlation* diversification measures, the *market portfolio* diversification measures and the *risk contribution* diversification measures.

1.1.1 Law of Large Numbers Diversification Measures

This category includes measures designed to capture the effect of law of large numbers diversification. This diversification strategy involves investing a small fraction of wealth in each of a large number of assets. A specific example is *naive diversification* (or equal weight portfolio), in which the same amount of wealth is invested in each available asset. Examples of law of large numbers (and in particular, naive diversification) measures are the effective number of constituents (see Carli et al., 2014; Deguest et al., 2013) and Bouchaud et al.'s (1997) class of measures which includes the Shannon and Gini-Simpson indexes (Zhou et al., 2013). Other examples of naive diversification measures can be found in Yu et al. (2014) and Lhabitant (2017).

1.1.2 Correlation Diversification Measures

This category includes measures designed to capture the effect of *correlation*[1] diversification. This diversification strategy is at the core of most of the decision theories including the expected utility theory and Yaari's (1987) dual theories, and is similar to the notion of *correlation aversion* (see Epstein and Tanny, 1980; Richard, 1975). Consequently, it can be viewed as the *rational* diversification principle for risk averse investors. It exploits interdependence between asset returns to reduce portfolio risk. The idea is that fewer assets are positively correlated, so the likelihood they do poorly at the same time in the same proportion is low and the protection offered by multilateral insurance, which is diversification, is better. Therefore, when there is correlation, it becomes dangerous to use law of large numbers diversification. *Correlation* diversification is recommended in Basel II (Committee of European Banking Supervisors, 2010) and Basel III (Basel Committee on Banking Supervision, 2010, 2013), and in Solvency II for calculating the solvency capital requirement (Committee of European Insurance and Occupational Pensions Supervisors, 2010a,b). Examples of *correlation* diversification measures are Embrechts et al.'s (1999) class of measures, Tasche's (2006) class of measures, the diversification ratio of Choueifaty and Coignard (2008), the diversification return of Booth and Fama (1992), the excess growth rate of Fernholz (2010), the return gap of Statman and Scheid (2005), the Goetzmann and Kumar's (2008) measure of diversification and the diversification delta of Vermorken et al. (2012).

1.1.3 Market Portfolio Diversification Measures

This category includes measures designed to capture the effect of market portfolio diversification. This diversification strategy was introduced by Sharpe (1964) and consists of holding a market portfolio or a market capitalization-weighted portfolio. It focuses on idiosyncratic risk reduction, so it fails during systematic crashes like the 2007-2009 financial crisis. Examples of market portfolio diversification measures are *portfolio size* (see Evans and Archer, 1968), Sharpe (1972)'s measure and Barnea and Logue (1973)'s measures.

1.1.4 Risk Contribution Diversification Measures

The last category includes measures designed to capture the effect of *risk contribution* diversification. This diversification strategy, also known as risk parity, became popular after the 2007-2009 financial crisis (Maillard et al., 2010; Qian, 2006). It consists of allocating portfolio risk equally among its components. Examples of risk contribution diversification measures are the *effective number of correlated bets* (see Carli et al., 2014; Roncalli, 2014) and the *effective number of uncorrelated bets* of Meucci (2009) and Meucci et al. (2014).

[1] The term *correlation* here refers here to any dependence measure including dissimilarity or similarity measures.

1.2 Towards a Rigorous Theory of Correlation Diversification Measures

We focus on the rich choice set of *correlation* diversification measures. A completely rigorous formulation of *correlation* diversification measurement is still lacking in the literature. None of the existing measures truly has theoretical foundations (axiomatic or decision-theoretic foundations). Although *correlation* diversification is at the core of most of the decision theories and is recommended by international regulatory agencies, no attention has been given to the conceptual problems involved in its measurement. Much of the academic literature on the theoretical foundations of risk management has been focused on the study of risk measurement (see Artzner et al., 1999; Follmer and Schied, 2002; Frittelli and Gianin, 2002, 2005; Rockafellar et al., 2006). Unfortunately, even if *correlation* diversification is taken into account in the standard risk measurement frameworks through the properties of *convexity*, *sub-additivity*, *comonotonic additivity*, *homogeneity* and *non-additivity for independence*, these risk measures do not quantify the *correlation* diversification effect properly. The reason is that risk reduction is not equivalent to diversification. Diversification implies risk reduction, but the reverse is not true, because risk can also be reduced by concentration. For example, in Artzner et al.'s (1999) and Föllmer and Weber's (2015) monetary risk measure theories, the possibility of reducing risks by concentration is taken into account through the property of *monotonicity*, and has the same importance as diversification. Consequently, standard risk measurement frameworks fail to adequately quantify and manage *correlation* diversification, except in the extreme case where all assets have the same risk.

The lack of rigorous theories of *correlation* diversification measures when the decision maker is risk averse does not favor *(i)* a rapid improvement in understanding the concept of diversification, *(ii)* a development of *coherent* measures, and *(iii)* a comparison of existing measures. The 2007-2008 crisis revealed that the concept of *correlation* diversification is misunderstood (Ilmanen and Kizer, 2012; Miccolis and Goodman, 2012; Statman, 2013). An example of the development of an inadequate *correlation* diversification measure is the *diversification delta* introduced in Vermorken et al. (2012) and revised in Flores et al. (2017).

Our paper is a first step towards a rigorous theory of *correlation* diversification measures. We provide an axiomatic foundation of the measurement of *correlation* diversification in a one-period portfolio theory under the assumption that the investor has complete information about the joint distribution of asset returns.

Specifically, in Section 3, we present and discuss a set of minimum desirable axioms that a measure of portfolio diversification must satisfy in order to be considered coherent.

In Section 4, we provide decision-theoretic foundations of our axioms by studying their rationality with respect to the two important decision theories under risk. The first is the classical expected utility theory. The second is Yaari's (1987) dual theory. More specifically, for each framework, we examine the compatibility of our axioms with investors' preference for diversification (PFD) by using the notion of PFD introduced by Dekel (1989) and extended later by Chateauneuf and Tallon (2002) and Chateauneuf and Lakhnati (2007).

We proceed as follows. First, using the notion of PFD, we identify the measure of portfolio diversification at the core of each theory. Next, we test the identified measure against our axioms. If the identified measure satisfies our axioms, we consider our axioms rationalized by the theory. In doing so, we show that our axioms are rationalized by *(a)* the expected utility theory if and only if one of the following conditions is satisfied: *(i)* risk is small in the sense of Pratt (1964) and absolute risk aversion is constant (see Proposition 1), or *(ii)* each distribution of asset returns belongs to the location-scale family and the certainty equivalent has a particular additive-separable form (see Proposition 2); and *(b)* Yaari's (1987) dual theory if and only if its probability distortion function is convex. These results strengthen the desirability, reasonableness and relevance of our axioms.

In Section 5, we examine a list of some of the most frequently used methods for measuring *correlation* diversification in terms of the axioms. This list includes:

(i) portfolio variance, a risk measure following the mean-variance model, often used to capture the benefit of diversification (Markowitz, 1952, 1959; Sharpe, 1964) and formally analyzed in Frahm and Wiechers (2013) as a measure of portfolio diversification;

(ii) the *diversification ratio* designed by Choueifaty and Coignard (2008), and used by the firm TOBAM[2] to manage billions in assets via its Anti-Benchmark® strategies in Equities and Fixed Income;

(iii) Embrechts et al.'s (1999) class of measures and its normalized version analyzed by Tasche (2006), which are widely used to quantify diversification in both the finance and insurance industries (see Bignozzi et al., 2016; Dhaene et al., 2009; Embrechts et al., 2013, 2015; Tong et al., 2012; Wang et al., 2015) and are recommended implicitly in some international regulatory frameworks (see Basel Committee on Banking Supervision, 2010; Committee on Risk Management and Capital Requirements, 2016).

We show that portfolio variance satisfies our axioms, but under the very restrictive (if not impossible) conditions that assets have identical variances (see part (i) of Proposition 4). This result, rather than weakening our axioms, reveals the limits of portfolio variance as an adequate measure of diversification in the mean-variance model.

We also show that the *diversification ratio* satisfies our axioms (see part (ii) of Proposition 4), and that Embrechts et al.'s (1999) (see part (iii) of Proposition 4) and Tasche's (2006) (see part (iv) of Proposition 4) classes of diversification measures satisfy our axioms, but under the condition that the underlying risk measure is convex (or quasi-convex),

[2] https://www.tobam.fr/

homogeneous, translation invariant and *reverse lower comonotonic additive*, which means that if the benefit of diversification is exhausted, then risks have lower comonotonicity (see item (xi) on page 8). These findings constitute supplementary evidence for the desirability, reasonableness and relevance of our axioms. They also show that measures such as the *diversification ratio*, Embrechts et al.'s (1999) and Tasche's (2006) classes of diversification measures can be justified by our axioms.

Our findings (parts (iii) and (iv) of Proposition 4) also establish the conditions under which a coherent risk measure in the sense of Artzner et al. (1999) induces a coherent diversification measure (see Corollary 1). This condition is the reverse lower comonotonic additive property. The expected shortfall (in the case of continuous distribution), which is chosen over Value-at-Risk in Basel III (see Basel Committee on Banking Supervision, 2013), and the concave distortion risk measures (see Sereda et al., 2010) induce coherent diversification measures. Our findings (parts (iii) and (iv) of Proposition 4) also imply that the deviation risk measure (see Rockafellar et al., 2006) induces a coherent diversification measure, but the family of convex risk measures (see Follmer and Schied, 2002; Frittelli and Gianin, 2002, 2005) does not. Our findings (part ((iv) of Proposition 4 in particular) also support the findings of Flores et al. (2017) that the *diversification delta* of Vermorken et al. (2012) is an inadequate measure of portfolio diversification.

Finally, in Section 6, we investigate the structure of representation of our axioms. We show that our axioms imply a family of representations, but this family is not unique. We provide some examples and one counterexample of this family of representations to support our argument.

Section 7 concludes the paper. Proofs are given in the appendix. Throughout the paper, vectors and matrices have bold style.

1.3 Related Literature

As mentioned above, the literature has focused exclusively on the design of *correlation* diversification measures. Some studies have presented and discussed desirable axioms to support their proposed measures. For example, Choueifaty et al. (2013) introduce the axiom of *duplication invariance* to support the *diversification ratio*. Evans and Archer (1968), Rudin and Morgan (2006) and Vermorken et al. (2012) present the *monotonicity in portfolio size* axiom to support portfolio size, the *portfolio diversification index* and the *diversification delta*, respectively. In addition to the axioms of *duplication invariance* and *monotonicity in portfolio size*, Carmichael et al. (2015) discuss the axioms of *degeneracy in portfolio size* and *degeneracy relative to dissimilarity* to support *Rao's Quadratic Entropy*. Meucci et al. (2014, Example 1, pp 4) discuss the axiom of *market homogeneity* to support the *effective number of bets*. Our research contributes to this literature by generalizing, completing and rationalizing this list of axioms to obtain a coherent axiomatic system.

In a recent contribution, De Giorgi and Mahmoud (2016b) develop an axiomatic structure for a diversification measure that is designed to capture the effect of naive diversification. By contrast, our axiomatic system is relevant for measures based on the notion of *correlation* diversification, assuming that the risk averse decision maker has complete information about the joint distribution of asset returns. Our work complements De Giorgi and Mahmoud (2016b) but it differs on an important point: their axiomatic system has a unique representative form and ours does not.

Finally, our work complements that of Artzner et al. (1999) on risk measurement. Whereas Artzner et al. (1999) provide a coherent axiomatic system of risk measures, our work provides a coherent axiomatic system of *correlation* diversification measures. Our work differs from that of Artzner et al. (1999) in two other important respects. First and foremost, we study, in Section 4, the rationality of our axiomatic system with respect to the expected utility (Propositions 1 and 2) and Yaari's (1987) dual (Propositions 3) theories. Second, our axiomatic system does not imply a unique family of representations.

2 Preliminaries

We consider a one-period model, so time diversification is impossible. We assume that the investor is risk averse and the investment opportunity set is a universe $\mathcal{A} = \{A_i\}_{i \in \mathcal{I}_N}$ of N assets (risky or not), where A_i denotes asset i of $\mathcal{A}$, $\mathcal{I}_N = \{1, ..., N\}$ is an index set and N is a strictly positive integer ($N \geq 1$). We also assume that short sales are restricted and we denote by $\mathbb{W} = \left\{\mathbf{w} = (w_1, ..., w_N)^\top \in \mathbb{R}_+^N : \sum_{i=1}^N w_i = 1\right\}$ the set of long-only portfolios associated with $\mathcal{A}$, where w_i is the weight of asset i in portfolio $\mathbf{w}$, $\top$ is a transpose operator and $\mathbb{R}_+$ is the set of positive real numbers. Our findings remain valid when short sales in the sense of Lintner (1965) are allowed, i.e. when the set of long/short portfolios is defined as $\mathbb{W}^- = \left\{\mathbf{w} = (w_1, ..., w_N)^\top \in \mathbb{R}^N : \sum_{i=1}^N |w_i| = 1\right\}$ with $\mathbb{R}$ the set of real numbers and $|.|$ the absolute value operator. A single-asset i portfolio is denoted $\boldsymbol{\delta}_i = (\delta_{i1}, ..., \delta_{iN})^\top$, where δ_{ij} is the Kronecker delta i.e. $\delta_{ii} = 1$ for each $i \in \mathcal{I}_N$ and $\delta_{ij} = 0$ for $i \neq j$, $i, j \in \mathcal{I}_N$. A portfolio that holds at least two assets is considered a diversified portfolio, while a portfolio that maximizes or minimizes a portfolio diversification measure is a well-diversified portfolio.

$R_i \in \mathcal{R}$ denotes the future return of asset i, where $\mathcal{R} = \mathbb{L}^\infty(\Omega, \mathcal{E}, P)$ is the vector space of bounded real-valued random variables on a probability space $(\Omega, \mathcal{E}, P)$, where Ω is the set of states of nature, $\mathcal{E}$ is the $\sigma-$ algebra of events, and P is a $\sigma-$ additive probability measure on $(\Omega, \mathcal{E})$. We assume that the investor has complete information about the joint distribution of $\mathbf{R} = (R_1, ..., R_N)^\top$. The expected value of R_i is $\mu_i = \mathrm{E}(R_i)$, its variance $\sigma_i^2 = \mathrm{Var}(R_i)$, its cumulative function $F_{R_i}(r_i)$, and its decumulative (survival) function $\overline{F}_{R_i}(r_i) = 1 - F_{R_i}(r_i)$, where $\mathrm{E}(.)$ and $\mathrm{Var}(.)$ are the operators of expectation and variance, respectively. The covariance between R_i and R_j is σ_{ij} and the covariance matrix is $\boldsymbol{\Sigma} = (\sigma_{ij})_{i,j=1}^N$. The Pearson's correlation between R_i and R_j is $\rho_{ij} = \frac{\sigma_{ij}}{\sigma_i \sigma_j}$ and the correlation matrix is $\boldsymbol{\rho} = (\rho_{ij})_{i,j=1}^N$. The vector of asset volatility is denoted $\boldsymbol{\sigma} = (\sigma_1, ..., \sigma_N)^\top$. The future return of portfolio $\mathbf{w}$ is $R(\mathbf{w}) = \mathbf{w}^\top \mathbf{R}$. Its expected value is $\mu(\mathbf{w}) = \mathbf{w}^\top \boldsymbol{\mu}$ and its variance $\sigma^2(\mathbf{w}) = \mathbf{w}^\top \boldsymbol{\Sigma}\, \mathbf{w}$. The cumulative and decumulative functions of $\mathbf{R}$ are $F_{\mathbf{R}}(\mathbf{r})$ and $\overline{F}_{\mathbf{R}}(\mathbf{r})$, respectively. When necessary, the subscript i will be replaced by A_i, $\mathbb{W}$ will be denoted $\mathbb{W}^N$ or $\mathbb{W}_{\mathcal{A}}^N$ and $\mathbf{w}$ will be denoted $\mathbf{w}_{\mathcal{A}}$.

Let ϱ denote a risk measure on $\mathcal{R}$.[3] More formally, ϱ is a mapping defined from $\mathcal{R}$ into $\mathbb{R}$, which can have the following desirable properties:

(i) Monotonicity: for all $X, Y \in \mathcal{R}$, if $X \leq Y$, then $\varrho(X) \geq \varrho(Y)$;

(ii) Sub-additivity: for all $X, Y \in \mathcal{R}$, $\varrho(X+Y) \leq \varrho(X) + \varrho(Y)$;

(iii) Convexity: for all $X, Y \in \mathcal{R}$, $\lambda \in [0,1]$, $\varrho(\lambda X + (1-\lambda)Y) \leq \lambda\, \varrho(X) + (1-\lambda)\, \varrho(Y)$;

(iv) Quasi-convexity: for all $X, Y \in \mathcal{R}$, $\lambda \in [0,1]$, $\varrho(\lambda X + (1-\lambda)Y) \leq \max(\varrho(X), \varrho(Y))$;

(v) Comonotonic additivity: for comonotonic $X, Y \in \mathcal{R}$, $\varrho(X+Y) = \varrho(X) + \varrho(Y)$;

(vi) Non-additivity for independence: for independent $X, Y \in \mathcal{R}$, $\varrho(X+Y) \neq \varrho(X) + \varrho(Y)$;

(vii) Translation invariance: for all $a \in \mathbb{R}$, $X \in \mathcal{R}$ and $\eta \geq 0$, $\varrho(X+a) = \varrho(X) - \eta\, a$;

(viii) Positive homogeneity: for all $\kappa \in \mathbb{R}$, $X \in \mathcal{R}$ and $b \geq 0$, $\varrho(b\,X) = b^\kappa \varrho(X)$;

(ix) Law invariance: if X, Y are identically distributed, denoted by $X = Y$, then $\varrho(X) \leq \varrho(Y)$.

(x) Positivity: for all nonconstant X, $\varrho(X) > 0$ and for all constant X, $\varrho(X) = 0$.

(xi) Reverse lower comonotonic additivity: for $X_i \in \mathcal{R}$, $i = 1, ..., N$, $\varrho\left(\sum_{i=1}^N X_i\right) = \sum_{i=1}^N \varrho(X_i)$ implies that the sequence $X_1, ..., X_N$ is lower comonotonic.

A random vector $\mathbf{X}$ is comonotonic if and only if there are non-decreasing functions f_i, $i \in \mathcal{I}_N$ and a random variable X such that $\mathbf{X} \overset{d}{=} (f_1(X), ..., f_N(X))$, where $\overset{d}{=}$ stands for "equally distributed" (see Dhaene et al., 2008). Intuitively, the comonotonicity corresponds to an extreme form of positive dependency. All returns are driven linearly or nonlinearly by a unique factor, but positively. For more details about the concept of comonotonicity and its applications in finance, we refer readers to Dhaene et al. (2002b) and Dhaene et al. (2002a).

In a real world environment, asset returns are usually not comonotonic, but can be comonotonic in the tails, as observed during the 2007-2009 financial crisis. The concept of upper comonotonicity was introduced and investigated in Cheung (2009). A random vector $\mathbf{X}$ is upper comonotonic if and only if $\mathbf{X}$ exhibits a comotonicity behavior in the upper tail. The lower comonotonicity is the opposite: a random vector $\mathbf{X}$ is lower comonotonic if and only if its opposite $-\mathbf{X}$ is upper comonotonic. We refer readers to Nam et al. (2011), Dong et al. (2010) and Hua and Joe (2012) for more details on the concept of upper comonotonicity.

The property of monotonicity is a natural requirement for a reasonable risk measure. The properties of sub-additivity, convexity, quasi-convexity, comonotonic additivity and non-

[3] Note that $\varrho(.) = \mathrm{Var}(.)$ in the case of the variance risk measure.

additivity for independence capture the diversification effect. The property of sub-additivity states that "a merger does not create extra risk" (Artzner et al., 1999). The properties of convexity and quasi-convexity imply that diversification should not increase risk. The property of comonotonic additivity implies that there is no benefit in terms of risk reduction to diversify across comonotonic risks. The property of non-additivity for independence rules out the possibility that the pooling of independent risks does not have a diversification effect. The property of translation invariance states that risk can be reduced by adding cash, except in the case where $\eta = 0$. The property of law invariance states that a risk measure $\varrho(X)$ depends only on the distribution of X i.e. $\varrho(X) = \varrho(F_X)$. The property of positive homogeneity states that a linear increase of the return by a positive factor leads to a non-linear increase in risk, except in the case where $\kappa = 1$. The property of positivity captures the idea that $\varrho(.)$ measures the degree of uncertainty in X. The property of reverse lower comonotonic additivity requires that if $\varrho\left(\sum_{i=1}^{N} X_i\right)$ is additive, then risks $X_1, ..., X_N$ are lower comonotonic. In other words, if the benefit of diversification is exhausted, then risks are lower comonotonic.

$\varrho(.)$ is called a coherent risk measure (see Artzner et al., 1999) if it satisfies the properties of monotonicity, sub-additivity, translation invariance and positive homogeneity (with $\kappa = 1$). $\varrho(.)$ is called a convex risk measure (see Follmer and Schied, 2002; Frittelli and Gianin, 2002, 2005), if it satisfies the properties of monotonicity, translation invariance and convexity. $\varrho(.)$ is called a deviation risk measure (see Rockafellar et al., 2006), if it satisfies the properties of sub-additivity, translation invariance ($\eta = 0$), positive homogeneity (with $\kappa = 1$) and positivity. For more details about these properties and for other desirable properties, readers are referred to Pedersen and Satchell (1998), Artzner et al. (1999), Song and Yan (2006), Song and Yan (2009), Sereda et al. (2010), Follmer and Schied (2010) and Wei et al. (2015).

3 Axioms

We present and discuss a set of minimum desirable axioms that obtain a measure of portfolio diversification that can be considered coherent. The next section provides the decision-theoretic foundations of these axioms.

Let us first introduce the definition of a portfolio diversification measure. How can we define a diversification measure of a portfolio $\mathbf{w}$? Although there is no unique definition of diversification in portfolio theory, the diversification interest variable and benefit, which is to say the distribution of portfolio weight $\mathbf{w}$ and risk or uncertainty reduction, respectively, are unique. Let Φ be a continuous measure of portfolio diversification. In the case where the investor is risk averse and has complete information about the joint distribution of asset returns $\mathbf{R}$, it is natural to represent Φ as a mapping from $\mathbb{W}$ into $\mathbb{R}$ conditional to $\mathbf{R}$

explicitly or implicitly; formally

$$\Phi \colon \mathbb{W} \to \mathbb{R} \tag{1}$$

$$\mathbf{w} \mapsto \Phi(\mathbf{w}|\mathbf{R}). \tag{2}$$

The form of the function $\Phi(.|\mathbf{R})$ depends on some properties of the portfolio diversification measure.

In the case where the investor has no information about $\mathbf{R}$, Φ can also be represented as a function of $\mathbf{w}$ i.e. $\Phi(\mathbf{w})$, and used to capture the effect of naive diversification.

Let now introduce our set of desirable axioms. There is no loss of generality in assuming that the well-diversified portfolio of $\Phi(\mathbf{w}|\mathbf{R})$, denoted $\mathbf{w}^*$, is obtained by maximization i.e.

$$\mathbf{w}^* \in \arg\underset{\mathbf{w}\in\mathbb{W}}{\mathrm{Max}}\ \Phi(\mathbf{w}|\mathbf{R}).$$

Therefore, given a measure Φ, we say that "portfolio $\mathbf{w}_1$ is more diversified than portfolio $\mathbf{w}_2$" if $\Phi(\mathbf{w}_1) > \Phi(\mathbf{w}_2)$.

Our first axiom formalizes investors' preference for diversification over $\mathbb{W}$. This axiom was first formulated in Carmichael et al. (2015) and is expressed in

Concavity (C). For each $\mathbf{w}_1$ and $\mathbf{w}_2 \in \mathbb{W}$, $\alpha \in [0,1]$ and $\mathbf{R} \in \mathcal{R}^N$,

$$\Phi(\alpha\,\mathbf{w}_1 + (1-\alpha)\,\mathbf{w}_2|\mathbf{R}) \geq \alpha\,\Phi(\mathbf{w}_1|\mathbf{R}) + (1-\alpha)\,\Phi(\mathbf{w}_2|\mathbf{R}) \tag{3}$$

and strict inequality for at least one α.

Concavity implies that holding different assets increases total diversification. It also ensures that the diversification is always beneficial and can be decomposed across asset classes.

Concavity can be replaced by a less restrictive axiom.

Quasi-Concavity (QC). For each $\mathbf{w}_1$ and $\mathbf{w}_2 \in \mathbb{W}$, $\alpha \in [0,1]$ and $\mathbf{R} \in \mathcal{R}^N$,

$$\Phi(\alpha\,\mathbf{w}_1 + (1-\alpha)\,\mathbf{w}_2|\mathbf{R}) \geq \min(\Phi(\mathbf{w}_1|\mathbf{R}), \Phi(\mathbf{w}_2|\mathbf{R})) \tag{4}$$

and strict inequality for at least one α.

Our next axiom is complementary to Concavity. It is favored by Carmichael et al. (2015) and is expressed in

Size Degeneracy (SD). There is a constant (for a normalization) $\underline{\Phi} \in \mathbb{R}$ such that for each $\mathbf{R} \in \mathcal{R}^N$,

$$\Phi(\boldsymbol{\delta}_i|\mathbf{R}) = \underline{\Phi} \text{ for each } i \in \mathcal{I}_N. \tag{5}$$

It states that all single-asset portfolios have the same degree of diversification and are the least diversified portfolio. Concavity and Size Degeneracy taken together imply that diversification is always better than full concentration or specialization; formally for each $i \in \mathcal{I}_N$, $\Phi(\boldsymbol{\delta}_i|\mathbf{R}) \le \Phi(\mathbf{w}|\mathbf{R})$. Size Degeneracy is clearly necessary to prevent portfolio concentration to remain undetected.

Our next axiom formalizes the behavior of $\Phi(\mathbf{w}|\mathbf{R})$ when $\mathbf{R}$ is homogeneous in the sense of perfect similarity. It is expressed in

Risk Degeneracy (RD). Let $\mathcal{A} = \{A_i\}_{i\in\mathcal{I}_N}$ be a universe of N assets such that $A_i = A$, for each $i \in \mathcal{I}_N$. Then, for each $\mathbf{w} \in \mathbb{W}$

$$\Phi(\mathbf{w}|\mathbf{R}) = \underline{\Phi}. \tag{6}$$

Risk Degeneracy ensures that there is no benefit to diversifying across perfectly similar assets. Such diversification is equivalent to full concentration. Risk Degeneracy is also necessary to keep portfolio concentration from going undetected. Risk Degeneracy generalizes Carmichael et al.'s (2015) axiom of *degeneracy relative to dissimilarity*, which states, "a portfolio formed solely with perfect similar assets must have the lowest diversification degree."

Our next axiom is complementary to Risk Degeneracy and is expressed in

Reverse Risk Degeneracy (RRD). Consider the equation $\Phi(\mathbf{w}|\mathbf{R}) = \underline{\Phi}$ on $\mathcal{R}$ for each $\mathbf{w} \in \mathbb{W}$ such that $\mathbf{w} \neq \boldsymbol{\delta}_i$, $i \in \mathcal{I}_N$ and without loss of generality assume that $w_i > 0$, for each $i \in \mathcal{I}_N$. Assume that a solution exists and is $\mathbf{R}^*$. Then $\mathbf{R}^*$ must be lower comonotonic. Note that $\mathbf{R}^*$ can be different from $\mathbf{R}$.

Reverse Risk Degeneracy is also necessary to prevent undetected portfolio concentration. It states that when $\Phi(\mathbf{w}|\mathbf{R}) = \underline{\Phi}$ with $\mathbf{w}$ a diversified portfolio, $\mathbf{R}_{\mathcal{I}_+} = (R_i)_{i\in\mathcal{I}_+}$ or its transformation is necessarily lower comonotonic, where $\mathcal{I}_+ = \{i|w_i > 0\}$. The following example is provided to get more of a sense of the importance of Reverse Risk Degeneracy.

Example 1 (Embrechts et al.'s (2009) Class of Diversification Measures). Consider Embrechts et al.'s (2009) class of diversification measures (see item (itememb) in Section 5) when the risk measure $\varrho(.)$ is additive for independence i.e for independent $X, Y \in \mathcal{R}$, $\varrho(X+Y) = \varrho(X) + \varrho(Y)$. This is the case when $\varrho(.)$ is the *mixed Esscher premium* or the *mixed exponential premium* analyzed in Goovaerts et al. (2004). In that case, according to Embrechts et al.'s (2009) class of diversification measures, any portfolio with assets with independent returns and single-asset portfolios would have the same degree of diversification, which is counterintuitive. Reverse Risk Degeneracy rules out this sub-class of Embrechts et al.'s (2009) and Tasche's (2006) diversification measures.

Our next axiom is the formalization of the property of *duplication invariance* of Choueifaty et al. (2013) analyzed in Carmichael et al. (2015). It is expressed in

Duplication Invariance (DI). Let $\mathcal{A}^{+} = \{A_i^{+}\}_{i \in \mathcal{I}_{N+1}}$ be a universe of assets such that $A_i^{+} = A_i$, for each $i \in \mathcal{I}_N$ and $A_{N+1}^{+} = A_k$, for $k \in \mathcal{I}_N$. Then

$$\Phi\big(\mathbf{w}_{\mathcal{A}}^{*}|\mathbf{R}_{\mathcal{A}}\big) = \Phi\big(\mathbf{w}_{\mathcal{A}^{+}}^{*}|\mathbf{R}_{\mathcal{A}^{+}}\big) \tag{7}$$

$$\mathbf{w}_{A_i}^{*} = \mathbf{w}_{A_i^{+}}^{*} \quad \text{for each } i \neq k,\ i \in \mathcal{I}_N \tag{8}$$

$$\mathbf{w}_{A_k}^{*} = \mathbf{w}_{A_k^{+}}^{*} + \mathbf{w}_{A_{N+1}^{+}}^{*}. \tag{9}$$

The reasonableness and relevance of Duplication Invariance is evident. It allows us to avoid risk concentration by ensuring that the optimal diversified portfolio is not biased towards multiple representative assets. It is necessary to prevent portfolio concentration from going undetected. The following example is provided to demonstrate the importance of Duplication Invariance.

Example 2 (Case $N = 2$). Consider the case where $N = 2$. In that case $\mathcal{A} = \{A_1, A_2\}$ and $\mathcal{A}^{+} = \{A_1, A_2, A_3^{+}\}$ such that $A_3^{+} = A_1$. Duplication Invariance states that the degree of diversification of $\mathcal{A}$ and $\mathcal{A}^{+}$ must be equal and optimally the weight of A_2 in $\mathcal{A}$ must be equal to the sum of the weights of A_2 and A_3^{+} in $\mathcal{A}^{+}$.

Our next axiom formalizes the relationship between diversification and portfolio size. It is expressed in

Size Monotonicity (M). Let $\mathcal{A}^{++} = \{A_i^{++}\}_{i \in \mathcal{I}_{N+1}}$ be a universe of assets such that $A_i^{++} = A_i$, for each $i \in \mathcal{I}_N$ and $A_{N+1}^{++} \neq A_i$, for each $i \in \mathcal{I}_N$. Then

$$\Phi\big(\mathbf{w}_{\mathcal{A}^{++}}^{*}|\mathbf{R}_{\mathcal{A}^{++}}\big) \geq \Phi\big(\mathbf{w}_{\mathcal{A}}^{*}|\mathbf{R}_{\mathcal{A}}\big). \tag{10}$$

Size Monotonicity is natural in portfolio diversification literature (see Carmichael et al., 2015; Evans and Archer, 1968; Rudin and Morgan, 2006; Vermorken et al., 2012). It reveals that increasing portfolio size does not decrease the degree of portfolio diversification. It also states that increasing portfolio size does not systematically increase the degree of portfolio diversification.

Our next axiom is an adaptation of the risk measure translation invariance axiom. It is expressed in

Translation Invariance (TI). Let $\mathcal{A}+a = \{A_i + a\}_{i \in \mathcal{I}_N}$ be a universe of assets such that $R_{A_i+a} = R_{A_i} + a$, for each $i \in \mathcal{I}_N$, $a \in \mathbb{R}$. Then for each $\mathbf{w} \in \mathbb{W}_{\mathcal{A}} = \mathbb{W}_{\mathcal{A}+a}$,

$$\Phi\left(\mathbf{w}|\mathbf{R}_{\mathcal{A}+a}\right) = \Phi\big(\mathbf{w}|\mathbf{R}_{\mathcal{A}}\big). \tag{11}$$

The desirability of Translation Invariance comes from the translation invariance risk measure axiom. It implies that adding the same amount of cash to asset returns does not change the degree of portfolio diversification. Consider the translation invariance risk measure axiom. Assume that $\eta = 0$. In that case, adding the same amount of cash to asset returns does not affect the degree of portfolio risk. The degree of portfolio diversification is not affected either. Now, assume that $\eta > 0$. In that case, adding the same amount of cash to asset returns reduces portfolio risk, but does not affect the degree of portfolio diversification.

However, when risk is defined as capital requirement or probability of loss (for example Expected Shortfall or Conditional Value-at-Risk), Translation Invariance can be seen as counterintuitive. To see this, consider the case where risk $\varrho(.)$ is defined as a capital requirement verifying the property of translation invariance. Assume that $a = \frac{\varrho(\mathbf{w}^\top \mathbf{R})}{\eta}$ with $\eta \neq 0$. Then $\varrho\left(\mathbf{w}^\top \mathbf{R} + a\right) = 0$, but $\Phi\left(\mathbf{w}|\mathbf{R}_{\mathcal{A}+a}\right) = \Phi\left(\mathbf{w}|\mathbf{R}_{\mathcal{A}}\right) \geq 0$. This counterintuitive result can be viewed as over diversification, and can be interpreted as an extreme precaution against extreme risk.

In the case where $\Phi(.|\mathbf{R})$ is a normalized measure, i.e. when $\Phi(\mathbf{w}|\mathbf{R})$ can be rewritten as follows

$$\Phi(\mathbf{w}|\mathbf{R}) = \frac{\widetilde{\Phi}(\mathbf{w}|\mathbf{R})}{\varrho(\mathbf{w}^\top \mathbf{R})}, \tag{12}$$

or equivalently

$$\Phi(\mathbf{w}|\mathbf{R}) = \frac{\widetilde{\Phi}(\mathbf{w}|\mathbf{R}) - \varrho(\mathbf{w}^\top \mathbf{R})}{\varrho(\mathbf{w}^\top \mathbf{R})}, \tag{13}$$

with $\widetilde{\Phi}(.|\mathbf{R})$ the portfolio diversification measure such that $\widetilde{\Phi}(\mathbf{w}|\mathbf{R} + a) = \widetilde{\Phi}(\mathbf{w}|\mathbf{R})$, the Translation Invariance must be replaced by with the following.

Translation Invariance-2 (TI2). Let $\mathcal{A} + a = \{A_i + a\}_{i \in \mathcal{I}_N}$ be a universe of assets such that $R_{A_i + a} = R_{A_i} + a$, for each $i \in \mathcal{I}_N$ with $a \in \mathbb{R}$. Then for each $\mathbf{w} \in \mathbb{W}_{\mathcal{A}+a}$

$$\frac{\partial \Phi\left(\mathbf{w}|\mathbf{R}_{\mathcal{A}+a}\right)}{\partial a} \geq 0, \tag{14}$$

$$\lim_{a \to -\infty} \Phi\left(\mathbf{w}|\mathbf{R}_{\mathcal{A}+a}\right) = \underline{\Phi}, \tag{15}$$

$$\lim_{a \to +\infty} \Phi\left(\mathbf{w}|\mathbf{R}_{\mathcal{A}+a}\right) = \underline{\Phi}, \tag{16}$$

$$\lim_{\substack{a \to \frac{\varrho\left(\mathbf{w}^\top \mathbf{R}_{\mathcal{A}}\right)}{\eta} \\ a > \frac{\varrho\left(\mathbf{w}^\top \mathbf{R}_{\mathcal{A}}\right)}{\eta}}} \Phi\left(\mathbf{w}|\mathbf{R}_{\mathcal{A}+a}\right) = -\infty, \tag{17}$$

$$\lim_{\substack{a \to \frac{\varrho\left(\mathbf{w}^\top \mathbf{R}_{\mathcal{A}}\right)}{\eta} \\ a < \frac{\varrho\left(\mathbf{w}^\top \mathbf{R}_{\mathcal{A}}\right)}{\eta}}} \Phi\left(\mathbf{w}|\mathbf{R}_{\mathcal{A}+a}\right) = \infty. \tag{18}$$

The idea behind Translation Invariance-2 is as follows. Equation (14) states that adding cash increases the diversification benefit. This is because adding cash reduces the total risk $\varrho(.)$ and does not affect diversification $\widetilde{\Phi}(.|\mathbf{R})$. Equations (15) and (16) ensure that when cash converges to $+\infty$ or $-\infty$, the diversification benefit vanishes because the whole system becomes homogeneous. Equations (17) and (18) capture the over diversification behavior of $\Phi(.|\mathbf{R})$ when risk converges to 0 and diversification becomes unnecessary.

Our next axiom is an adaptation of the positive homogeneity of risk measure axiom. It is expressed in

Homogeneity (H). Let $b\mathcal{A} = \{bA_i\}_{i\in\mathcal{I}_N}$ be a universe of assets such that $R_{bA_i} = b\,R_{A_i}$, for each $i \in \mathcal{I}_N$ with $b \geq 0$. Then there exists $\kappa \in \mathbb{R}$ such that for each $\mathbf{w} \in \mathbb{W}_{\mathcal{A}} = \mathbb{W}_{b\mathcal{A}}$

$$\Phi\big(\mathbf{w}|\mathbf{R}_{b\mathcal{A}}\big) = b^{\kappa}\Phi\big(\mathbf{w}|\mathbf{R}_{\mathcal{A}}\big). \tag{19}$$

The desirability of Homogeneity comes naturally from the homogeneous property of the risk measure. In the case where $\Phi(.|\mathbf{R})$ is a normalized measure, κ must be equal to zero, which ensures that $\Phi(.|\mathbf{R})$ must not depend on scalability.

Our last axiom presents the behavior of $\Phi(\mathbf{w}|\mathbf{R})$ when $R_1,...,R_N$ are exchangeable random variables. First, let us recall the definition of exchangeable random variables.

Definition 1 (Exchangeability). The random variables $R_1,...,R_N$ are said to be exchangeable if and only if their joint distribution $F_{\mathbf{R}}(\mathbf{r})$ is symmetric.

A well-known example of an exchangeable sequence of random variables is an independent and identically distributed sequence of random variables. For more details on exchangeable random variables, we refer readers to Aldous (1985).

Our last axiom is expressed in

Symmetry (S). If $R_1,...,R_N$ are exchangeable, then $\Phi(\mathbf{w}|\mathbf{R})$ is symmetric in $\mathbf{w}$.

Symmetry states that a portfolio diversification measure must be symmetric in $\mathbf{w}$ if $R_1,...,R_N$ are exchangeable. The thinking behind Symmetry is that the exchangeable random variables imply homogeneous risks. Thus, the decision maker must be indifferent in terms of diversification between $\mathbf{w}$ and $\mathbf{\Pi w}$, where $\mathbf{\Pi}$ is a permutation matrix.

From Marshall et al. (2011, C.2. and C.3. Propositions, pp 97-98), Symmetry and Concavity or Quasi-Concavity taken together imply that $\Phi(\mathbf{w}|\mathbf{R})$ is Schur-concave in $\mathbf{w}$ when $R_1,...,R_N$ are exchangeable. As a result, $\Phi(\mathbf{w}|\mathbf{R})$ must be a measure of naive diversification and the optimal diversified portfolio $\mathbf{w}^*$ must be the naive portfolio $\frac{\mathbf{1}}{N}$ when $R_1,...,R_N$ are exchangeable. This result is consistent with the principle that the exchangeability assumption on $R_1,...,R_N$ is equivalent to the assumption that the decision maker has no information about asset risk characteristics $\mathbf{R}$. Moreover, Symmetry and Concavity or Quasi-Concavity taken together imply the axiom of *market homogeneity* implicitly analyzed in Meucci et al. (2014, Example 1, pp 4).

4 Rationalization

This section studies the rationality of our axioms with respect to the two most important decision theories under risk: the expected utility theory and Yaari's (1987) dual theory. For each theory, we examine the compatibility of our axioms with investors' preference for diversification (PFD). We proceed as follows. First, from the notion of PFD, we identify the measure of portfolio diversification at the core of each model. Next, we test the identified measure against our axioms. If the identified measure satisfied our axioms, we consider that our axioms are rationalized by the model.

There are several notions of PFD in the theory of choice under risk or uncertainty (see De Giorgi and Mahmoud, 2016a). We consider the ideas introduced by Dekel (1989) and extended later by Chateauneuf and Tallon (2002) and Chateauneuf and Lakhnati (2007) to the space of random variables. Let $\succeq$ be the preference relation over $\mathcal{R}$ of a decision maker (i.e., an investor). The chosen notion of PFD is defined as follows.

Definition 2 (Chateauneuf and Tallon (2002)). The preference relation $\succeq$ exhibits preference for diversification if for any $R_i \in \mathcal{R}$ and $\alpha_i \in [0,1]$, $i \in \mathcal{I}_N$ such that $\sum_{i=1}^{N} \alpha_i = 1$,

$$R_1 \sim R_2 \sim \ldots \sim R_N \Rightarrow \sum_{i=1}^{N} \alpha_i R_i \succeq R_j \quad \text{for each } j \in \mathcal{I}_N. \tag{20}$$

Definition 2 states that if assets are equally desirable, then the investor will want to diversify. This notion of PFD is equivalent to the notion of risk aversion in the expected utility theory and implies the notion of strong risk aversion (i.e. risk aversion in the sense of mean preserving spread as defined in De Giorgi and Mahmoud (2016a, Definition 9, pp. 152)) in Yaari's (1987) dual theory.

4.1 Expected Utility Theory

Let us first study the rationality of our axioms with respect to the expected utility (EU) theory. Assume that $\succeq$ has an expected utility representation. Then

$$R_1 \succeq R_2 \Longleftrightarrow \mathrm{E}_u(R_1) \geq \mathrm{E}_u(R_2), \tag{21}$$

where $\mathrm{E}_u(R) = \mathrm{E}(u(R)) = \int u(r) dF_R(r)$ with $u : \mathcal{R} \longrightarrow \mathbb{R}$ is an increasing von Neumann-Morgenstern utility function for wealth. Moreover, $u(.)$ is unique up to positive affine transformations. The shape of $u(.)$ determines investors' risk attitude and diversification profile. The investors are risk averse, -neutral and -lover when $u(.)$ is concave, linear and convex respectively. The investors have a PFD only when $u(.)$ is concave i.e. when they are risk averse. In this paper, we assume that $u(.)$ is concave to be consistent with our hypothesis of risk averse investors and consequently with the notion of PFD.

Definition 2 is equivalent to the following.

Definition 3. The preference relation $\succeq$ shows a preference for diversification if for any $R_i \in \mathcal{R}$ and $\alpha_i \in [0,1]$, $i \in \mathcal{I}_N$ such that $\sum_{i=1}^{N} \alpha_i = 1$, the following equivalent conditions are satisfied

$$
\begin{aligned}
&(i)\ \mathrm{E}_u(R_1) = \ldots = \mathrm{E}_u(R_N) \Longrightarrow \mathrm{E}_u\left(\textstyle\sum_{i=1}^{N} \alpha_i\, R_i\right) \geq \mathrm{E}_u(R_j) \quad \text{for each } j \in \mathcal{I}_N \\
&(ii)\ \varrho_{C_u}(R_1) = \ldots = \varrho_{C_u}(R_N) \Longrightarrow \varrho_{C_u}\left(\textstyle\sum_{i=1}^{N} \alpha_i\, R_i\right) \leq \varrho_{C_u}(R_j) \quad \text{for each } j \in \mathcal{I}_N \\
&(iii)\ \varrho_{\pi_u}(R_1) = \ldots = \varrho_{\pi_u}(R_N) \Longrightarrow \varrho_{\pi_u}\left(\textstyle\sum_{i=1}^{N} \alpha_i\, R_i\right) \leq \varrho_{\pi_u}(R_j) \quad \text{for each } j \in \mathcal{I}_N,
\end{aligned}
$$

where $\varrho_{C_u}(R) = -C_u(R)$ is a risk measure induced by the certainty equivalent $C_u(R) = u^{-1}(\mathrm{E}_u(R))$ and $\varrho_{\pi_u}(R) = \pi_u(-R)$ is induced by the risk premium $\pi_u(R) = \mathrm{E}(R) - C_u(R)$ of $u(.)$.

Because diversification is a risk reduction tool, we focus on parts (ii) and (iii) of Definition 3. Multiplying the inequality in (ii) and (iii) by α_j and summing over j, we obtain

$$
\varrho_l\left(\sum_{i=1}^{N} \alpha_i\, R_i\right) \leq \sum_{i=1}^{N} \alpha_j\, \varrho_l(R_j) \quad \text{for each } l \in \{C_u,\, \pi_u\}. \tag{22}
$$

From (22), following Embrechts et al. (1999) (see item (iii) on page 20), the gain of diversification in the EU theory can be measured by the difference

$$
\underline{\varrho}_l\left(\mathbf{w}|\mathbf{R}\right) = \sum_{i=1}^{N} w_i\, \varrho_l(1 + R_i) - \varrho_l\left(1 + \sum_{i=1}^{N} w_i\, R_i\right) \quad \text{for each } l \in \{C_u,\, \pi_u\}. \tag{23}
$$

The definition of compatibility with the PFD in the EU theory is based on $\underline{\varrho}_l\left(\mathbf{w}|\mathbf{R}\right)$ for each $l \in \{C_u,\, \pi_u\}$ and is defined as follows:

Definition 4 (Compatibility with PFD in the EU theory). Our axioms are compatible with the PFD in the EU theory if and only if they are satisfied by $\underline{\varrho}_{C_u}\left(\mathbf{w}|\mathbf{R}\right)$ or $\underline{\varrho}_{\pi_u}\left(\mathbf{w}|\mathbf{R}\right)$.

Using Definition 4, we establish the necessary and sufficient conditions for the compatibility of our axioms with the PFD in the EU theory. Two cases are considered.

4.1.1 Case 1: Risk is Small

In this first case, we assume that risk is small in the sense of Pratt (1964) i.e. measured by σ_i^2, for each $i \in \mathcal{I}_N$. We refer to this compatibility as local compatibility. The following proposition establishes the necessary and sufficient conditions.

Proposition 1 (Local Compatibility with PFD in the EU Theory). *If risk is small, then our axioms are compatible with the PFD in the EU theory if and only if the absolute risk aversion of* $u(.)$ *is constant; formally,* $k(x) = -\frac{u''(x)}{u'(x)} = c$, $c \in \mathbb{R}$, *where* $u'(.)$ *and* $u''(.)$ *are the first and second derivatives of* $u(.)$.

Proposition 1 shows that our axioms can be rationalized by the EU theory if risk is small and the absolute risk aversion of $u(.)$ is constant. The negative exponential utility function, $u(x) = -\exp(-\lambda x)$ with $\lambda > 0$, the investor's risk aversion coefficient is the only example of a concave utility function that implies constant absolute risk aversion. Thus, if risk is small, our axioms can be rationalized by the EU theory if and only if $u(.)$ is the negative exponential utility.

4.1.2 Case 2: Location-Scale Family of Distributions

In the second case, we assume that each distribution of asset returns belongs to the location-scale family. This family of distributions includes, among others, the normal, student's t and all other elliptical distributions. For more details see Meyer et al. (1987). The following proposition establishes the necessary and sufficient conditions.

Proposition 2 (Compatibility with PFD in the EU Theory: location-scale family). *If each asset returns distribution belongs to the location-scale family, then our axioms are compatible with the PFD in the EU theory if and only if the certainty equivalent has the following additive separable form*

$$C_u(R) = \mu - g(\sigma) \tag{24}$$

for a strictly increasing continuous and homogeneous function $g(.)$ *on* $\mathbb{R}_+$.

Below, we present an example of location-scale distribution and utility function for which Proposition 2 is valid.

Example 3 (Normal Distribution and Negative Exponential Utility). Assume that the asset returns are normally distributed and the utility function is negative exponential. It is proven in the literature that

$$C_u(R(\mathbf{w})) = \mathbf{w}^\top \boldsymbol{\mu} - \lambda\sigma^2(\mathbf{w}). \tag{25}$$

Proposition 2 also implies that our axioms can also be rationalized by the additive separable mean-variance utility functions (including Markowitz (1952)'s mean-variance utility) axiomatized by Nakamura (2015, Theorem 4., pp. 544).

Propositions 1 and 2 jointly represent the necessary and sufficient conditions of our axioms to be compatible with the PFD in the EU theory. In Propositions 1 and 2 we have compatibility only when risk is measured by variance, so the conditions might be thought to be restrictive, thereby considerably weakening the desirability of our axioms. However, this is not the case, because the majority of our axioms remain compatible with the PFD in the EU theory when we consider other standard utility functions. For example, exploiting the results in Müller (2007), if we consider the negative exponential utility with a non-Location-Scale family of distributions, one can verify that $\underline{\varrho}_{C_u}(\mathbf{w}|\mathbf{R})$ satisfies all the axioms except Homogeneity. In the case of the power or logarithmic utility function, one can verify that $\underline{\varrho}_{C_u}(\mathbf{w}|\mathbf{R})$ satisfies all the axioms, except Concavity, Quasi-Concavity, Translation Invariance and Homogeneity. Second, as we show in the next subsection, our axioms are also relevant when risk is not completely captured by the variance.

In sum, Propositions 1 and 2 provide a decision-theoretic foundation of our axioms and consequently strengthen their desirability, reasonableness and relevance.

4.2 Yaari's (1987) Dual Theory

Despite the importance of the EU theory in the theory of rational choice under risk, it has been shown that it often fails to describe and predict peoples' choices properly (see Allais, 1953; Kahneman and Tversky, 1979). As a consequence, alternative theories of choice were proposed; see Schoemaker (1982), Machina (1987) and Starmer (2000) for comprehensive reviews. In this section, we study the rationality of our axioms with respect to one of the most successful of them: Yaari's (1987) dual (DU) theory of choice, which is a special case of the rank dependent utility theory of Quiggin (1982); see also Quiggin (2012).

Yaari's (1987) DU theory was constructed from the EU theory by replacing the independence axiom by the dual independence axiom, which states that for any $X, Y, Z \in \mathcal{R}$, if X is preferred to Y, then $\left(\alpha F_X^{-1} + (1-\alpha)F_Z^{-1}\right)^{-1}$ is preferred to $\left(\alpha F_Y^{-1} + (1-\alpha)F_Z^{-1}\right)^{-1}$. Doing so, Yaari (1987) obtained a preference functional that is linear concerning payoffs and nonlinear concerning probability, which is the opposite of the EU theory in which the preference functional is nonlinear concerning payoffs and linear concerning probability.

More formally, assume that $\geq$ has a DU theory representation. Then, from Yaari (1987) (see also Tsanakas and Desli, 2003),

$$R_i \geq R_j \Leftrightarrow \mathrm{E}_{\overline{h}}(R_i) \geq \mathrm{E}_{\overline{h}}(R_j), \tag{26}$$

where

$$\mathrm{E}_{\overline{h}}(R) = \int_{-\infty}^{0} \left(\overline{h}\left(\overline{F}_R(r)\right) - 1\right) dr + \int_{0}^{\infty} \overline{h}\left(\overline{F}_R(r)\right) dr = \int_{-\infty}^{\infty} r dh\left(F_R(r)\right) \tag{27}$$

with $\overline{h}, h : [0,1] \longmapsto [0,1]$ being increasing functions satisfying $\overline{h}(0) = h(0) = 0$ and $\overline{h}(1) = h(1) = 1$ such that $\overline{h}(u) = 1 - h(1-u)$. $\overline{h}(.)$ is called the probability distortion function and

$h(.)$ represents its dual.

Like in the EU theory, in the DU theory, the investor's risk profile can be characterized by some conditions on $\overline{h}(.)$. However, the notions of risk aversion are not equivalent in the DU theory. The investor is risk averse in the sense of $\mathrm{E}_{\overline{h}}(R) \leq \mathrm{E}_{\overline{h}}(\mathrm{E}(R)) = \mathrm{E}(R)$ if and only if $h(u) \leq u, \ \forall\, u \in [0,1]$. The investor is risk averse in the sense of mean preserving spread as defined in De Giorgi and Mahmoud (2016a, Definition 9, pp. 152) if and only if $\overline{h}(.)$ is convex and $\overline{h}(u) \neq u$ or $h(.)$ is concave and $h(u) \neq u$. In this paper, to be consistent both with our hypothesis of risk averse investors and our notion of PFD, we assume that $\overline{h}(.)$ is convex and $\overline{h}(u) \neq u$ or equivalently $h(.)$ is concave and $h(u) \neq u$.

Like in the EU theory, the gain of diversification in the DU theory can be measured by the difference

$$\underline{\varrho}_l\left(\mathbf{w}|\mathbf{R}\right) = \sum_{i=1}^{N} w_i \varrho_l(1+R_i) - \varrho_l\left(1+\sum_{i=1}^{N} w_i\, R_i\right) \text{ for each } l \in \{C_{\overline{h}},\, \pi_{\overline{h}}\}, \tag{28}$$

where $C_{\overline{h}}(.)$ and $\pi_{\overline{h}}(.)$ are respectively the certainty equivalent and the risk premium associated to the DU utility function $\mathrm{E}_{\overline{h}}(.)$. The certainty equivalent $C_{\overline{h}}(.)$ is $\mathrm{E}_{\overline{h}}(.)$ itself (see Yaari, 1987, pp. 101) and the risk premium is $\pi_{\overline{h}}(R) = \mathrm{E}(R) - \mathrm{E}_{\overline{h}}(R)$ (see Denuit et al., 1999). The risk premium can also be derived from $\mathrm{E}_{\overline{h}}(.)$ using the indifference arguments as in Denuit et al. (2006) and Tsanakas and Desli (2003). Formally, the risk premium $\varrho_{\pi_h}(R)$ is determined such that

$$\mathrm{E}_{\overline{h}}(r) = \mathrm{E}_{\overline{h}}(r - R + \varrho_{\pi_h}(R)). \tag{29}$$

From (29), one obtains

$$\pi_h(R) = -\mathrm{E}_{\overline{h}}(-R) = \mathrm{E}_h(R) = \int_{-\infty}^{0}\left(h\left(\overline{F}_R(r)\right)-1\right)dr + \int_{0}^{\infty} h\left(\overline{F}_R(x)\right)dr. \tag{30}$$

$\pi_h(.)$ is also known as the distortion risk measure and is equivalent to the spectral risk measure (see Gzyl and Mayoral, 2008; Sereda et al., 2010). $\varrho_{C_{\overline{h}}} = -C_{\overline{h}}(R)$ is the risk measure induced by $C_{\overline{h}}(.)$, $\varrho_{C_{\overline{h}}} = \pi_{\overline{h}}(-R)$ is the risk measure induced by $\pi_{\overline{h}}(.)$ and $\varrho_{\pi_h} = \pi_h(-R)$ is the risk measure induced by $\pi_h(R)$.

The definition of compatibility with the PFD in the DU theory is based on $\underline{\varrho}_l\left(\mathbf{w}|\mathbf{R}\right)$, for each $l \in \{C_{\overline{h}},\, \pi_{\overline{h}}\}$ and is defined as follows.

Definition 5 (Compatibility with PFD in the DU theory). Our axioms are compatible with the PFD in the DU theory if they are satisfied by $\underline{\varrho}_{C_{\overline{h}}}\left(\mathbf{w}|\mathbf{R}\right)$ or $\underline{\varrho}_{\pi_{\overline{h}}}\left(\mathbf{w}|\mathbf{R}\right)$.

Proposition 3 examines this compatibility.

Proposition 3 (Compatibility with PFD in the DU theory). *Our axioms are compatible with the PFD in the DU theory if and only if $\overline{h}(.)$ is convex or $h(.)$ is concave.*

Proposition 3 shows that our axioms can be rationalized by the DU theory of choice if and only if $\overline{h}(.)$ is convex or equivalently $h(.)$ is concave. This result provides another decision-theoretic foundation of our axioms and consequently strengthens their desirability, reasonableness and relevance. It also implies that any concave distortion risk measure induces a coherent diversification measure.

5 Existing Diversification Measures

In this section, we explore whether some useful methods of measuring *correlation* diversification satisfy our axioms. We consider the four *correlation* diversification measures used most frequently on the marketplace and by academic researchers in both finance and insurance:

(i) Portfolio variance

$$\sigma^2(\mathbf{w}|\mathbf{R}) = \mathbf{w}^\top \mathbf{\Sigma}\, \mathbf{w}.$$

(ii) Diversification ratio (DR)

$$\mathrm{DR}(\mathbf{w}|\mathbf{R}) = \frac{\mathbf{w}^\top \boldsymbol{\sigma}}{\sqrt{\mathbf{w}^\top \mathbf{\Sigma}\, \mathbf{w}}}.$$

(iii) Embrechts et al.'s (2009) class of measures

$$\mathrm{D}_\varrho(\mathbf{w}|\mathbf{R}) = \sum_{i=1}^{N} \varrho\left(w_i R_i\right) - \varrho\left(\mathbf{w}^\top \mathbf{R}\right).$$

(iv) Tasche's (2007) class of measures

$$\mathrm{DR}_\varrho(\mathbf{w}|\mathbf{R}) = \frac{\varrho\left(\mathbf{w}^\top \mathbf{R}\right)}{\sum_{i=1}^{N} \varrho(w_i R_i)}.$$

Portfolio variance is the risk measure in the mean-variance model. It is usually used to quantify the benefit of diversification (Markowitz, 1952, 1959; Sharpe, 1964), and is formally analyzed as a portfolio diversification measure in Frahm and Wiechers (2013).

The diversification ratio (DR) is a diversification measure introduced by Choueifaty and Coignard (2008); see also Choueifaty et al. (2013). An intuitive interpretation of the DR is the Sharpe ratio when each asset's volatility is proportional to its expected premium i.e. $\mathrm{E}(R_i) - R_N = \delta\sigma_i$, for each $i \in \mathcal{I}_{N-1}$ where $\delta > 0$ and R_N is the rate of the risk-free asset. DR is used in the finance industry by the french firm TOBAM to manage billions worth of assets via its Anti-Benchmark® strategies in Equities and Fixed Income.

Embrechts et al.'s (2009) class (D_ϱ) is the class of diversification measures induced by a risk measure $\varrho(.)$. An intuitive interpretation can be provided to D_ϱ when $\varrho(.)$ is homogeneous of degree one. In that case,

$$\mathrm{D}_\varrho(\mathbf{w}|\mathbf{R}) = \sum_{i=1}^{N} w_i \left(\varrho\left(R_i\right) - \varrho\left(\mathbf{w}^\top \mathbf{R}\right)\right).$$

The term in parentheses, $\varrho\left(R_i\right) - \varrho\left(\mathbf{w}^\top \mathbf{R}\right)$, measures the benefit of diversification, in terms of risk reduction, of holding portfolio $\mathbf{w}$ instead of concentrating on single-asset i. It follows that D_ϱ quantifies the average benefit of diversification. Tasche's (2007) class of diversification measures (DR_ϱ) is a normalized version of D_ϱ. Some authors (see Degen et al., 2010; Mao et al., 2012) refer to DR_ϱ as a measure of concentration risk and $1 - \mathrm{DR}_\varrho$ as a measure of diversification benefit. D_ϱ and DR_ϱ are the most commonly used in the finance and insurance literature (see Bignozzi et al., 2016; Dhaene et al., 2009; Embrechts et al., 2013, 2015; Tong et al., 2012; Wang et al., 2015) and recommended implicitly in some international regulatory frameworks (see Basel Committee on Banking Supervision, 2010; Committee on Risk Management and Capital Requirements, 2016).

The following proposition analyzes these four measures in the light of our axioms.

PROPOSITION 4. *The following statements hold.*

(i) Portfolio variance satisfies our axioms if and only if assets have the same standard deviation i.e $\sigma_i = \sigma_j,\ i, j = 1, ..., N$.
(ii) The diversification ratio satisfies our axioms.
(iii) Embrechts et al.'s (2009) class of measures satisfies our axioms if and only if $\varrho(.)$ *is convex (or quasi-convex), positively homogeneous, translation invariant and reverse lower comonotonic additive.*
(iv) Tasche's (2007) class of measures satisfies our axioms if and only if $\varrho(.)$ *is convex (or quasi-convex), positively homogeneous, translation invariant and reverse lower comonotonic additive.*

Part (i) of Propositions 4 shows that the portfolio variance satisfies our axioms, but under very restrictive (if not impossible) conditions that assets have identical variances. More specifically, portfolio variance satisfies our axioms, except Size Degeneracy, Risk Degeneracy and Reverse Risk Degeneracy. This result, rather than weakening our axioms, reveals the limits of portfolio variance as an adequate measure of diversification in the mean-variance model.

Part (ii) of Propositions 4 shows that the DR is coherent. Parts (iii) and (iv) show that Embrechts et al.'s (1999) and Tasche's (2006) classes of measures also are coherent, but under the same conditions that the risk measure $\varrho(.)$ is convex (or quasi-convex), homogeneous, translation invariant and reverse lower comonotonic additive.[4] These findings strengthen both the axioms and the measures. The axioms are applicable to a number of measures that we have considerable experience with. The measures have several properties whose desirability can be rationalized by the expected utility theory and Yaari's (1987) dual theory, and their popular use in empirical works and on the marketplace can be defended by our axioms.

Because Embrechts et al.'s (1999) and Tasche's (2006) classes of measures are diversification measures induced by risk measure, parts (iii) and (iv) also establish the conditions under which a coherent risk measure induces a coherent diversification measure.

Corollary 1 (Coherent risk measure (Artzner et al., 1999)). *A coherent risk measure induces a coherent diversification measure if and only if the coherent risk measure is reverse lower comonotonic additive.*

Corollary 1 follows from the fact that any coherent risk measure is convex, positively homogeneous (with $\kappa = 1$), translation invariant and all coherent risk measures are not reverse lower comonotonic additive. An example of a coherent risk measure that is not reverse lower comonotonic additive is the expectation risk measure i.e. $\varrho(X) = \mathrm{E}(X)$. An example of a coherent risk measure that is reverse lower comonotonic additive is any concave distortion risk measure as implied by Proposition 3. It follows that the expected shortfall, which is chosen over Value-at-Risk in Basel III (see Basel Committee on Banking Supervision, 2013), induces a coherent diversification measure in the case of continuous distribution.

Parts (iii) and (iv) of Propositions 4 also imply that the family of deviation risk measures of Rockafellar et al. (2006) induces a family of coherent diversification measures, but not the family of convex risk measures (see Follmer and Schied, 2002; Frittelli and Gianin, 2002, 2005).

Part (iv) of Propositions 4 also supports the findings of Flores et al. (2017) that the *diversification delta* of Vermorken et al. (2012) is an inadequate measure of portfolio diversification.

6 Representation

To close this paper, we examine whether or not our axioms imply a family of representations. As mentioned in Section 3, from Marshall et al. (2011, C.2. and C.3. Propositions, pp 97-98), Symmetry and Concavity or Quasi-Concavity taken together imply that $\Phi(\mathbf{w}|\mathbf{R})$ is Schur-concave in $\mathbf{w}$ when the sequence $R_1, ..., R_N$ is exchangeable. Therefore, from Marshall et al. (2011, B.1. Proposition, pp 393), $\Phi(\mathbf{w}|\mathbf{R})$ can have the following representation form

[4] Embrechts et al.'s (1999) and Tasche's (2006) classes of measures satisfy our axioms under the same conditions because the Tasche's (2006) class of measures is a normalized version of the Embrechts et al.'s (1999) class of measures.

$$\Phi(\mathbf{w}|\mathbf{R}) = \mathrm{E}\left(\phi(\mathbf{w},\mathbf{R})\right), \tag{31}$$

where $\phi(\mathbf{w},\mathbf{R})$ satisfies Size Degeneracy, Risk Degeneracy, Reverse Risk Degeneracy, Duplication Invariance, Size Monotonicity, Translation Invariance, Homogeneity, and the following additional properties

(i) $\phi(\mathbf{w},\mathbf{R})$ is concave in $\mathbf{w}$ for each fixed $\mathbf{R} \in \mathcal{R}^N$;
(ii) $\phi(\mathbf{\Pi}\mathbf{w},\mathbf{\Pi}\mathbf{R}) = \phi(\mathbf{w},\mathbf{R})$ for all permutations $\mathbf{\Pi}$;
(iii) $\phi(\mathbf{w},\mathbf{R})$ is Borel-measurable in $\mathbf{R}$ for each fixed $\mathbf{w}$.

The first two properties ensure that $\phi(\mathbf{w},\mathbf{R})$ is concave in $\mathbf{w}$ and symmetric in $\mathbf{w}$ when $\mathbf{R}$ is exchangeable, therefore satisfying Concavity and Symmetry. Below, we present two examples of $\phi(\mathbf{w},\mathbf{R})$.

Example 4 (Rao's Quadratic Entropy). Consider $\phi(\mathbf{w},\mathbf{R})$ such that

$$\phi(\mathbf{w},\mathbf{R}) = \sum_{i,j=1}^{N} |(R_i - \mu_i) - (R_j - \mu_j)|^q w_i w_j, \ 0 < q \le 2 \tag{32}$$

Obviously, $\phi(\mathbf{w},\mathbf{R})$ satisfies the three above properties and Size Degeneracy, Risk Degeneracy, Reverse Risk Degeneracy, Duplication Invariance, Size Monotonicity, Translation Invariance, and Homogeneity. As a consequence $\Phi(\mathbf{w}|\mathbf{R}) = \mathrm{E}\left(\phi(\mathbf{w},\mathbf{R})\right)$ is a coherent class of portfolio diversification measures analyzed in Carmichael et al. (2015) and Carmichael et al. (2018) under the name of *Rao's Quadratic Entropy.* In the case where $q = 2$, $\Phi(\mathbf{w}|\mathbf{R})$ coincides with the *diversification returns*, we see obtain the popular diversification measure analyzed in Willenbrock (2011), Chambers and Zdanowicz (2014), Bouchey et al. (2012), Qian (2012) and in Fernholz (2010) under the name *excess growth rate.*

Example 5 (Embrechts et al.'s (2009) Class of Diversification Measures). Consider $\phi(\mathbf{w},\mathbf{R})$ such that

$$\phi(\mathbf{w},\mathbf{R}) = \sum_{i,j=1}^{N} \max\left(w_i(-R_i - \mathrm{VaR}_\theta(-R_i)), 0\right) - \max\left(-\mathbf{w}^\top\mathbf{R} - \mathrm{VaR}_\theta\left(-\mathbf{w}^\top\mathbf{R}\right), 0\right) \ p \in (0,1), \tag{33}$$

where $\mathrm{VaR}_p(X)$ is the Value-at-risk of X at level p. As we can see, $\phi(\mathbf{w},\mathbf{R})$ satisfies the three above properties and Size Degeneracy, Risk Degeneracy, Reverse Risk Degeneracy, Duplication Invariance, Size Monotonicity, Translation Invariance, and Homogeneity. As a consequence $\Phi(\mathbf{w}|\mathbf{R}) = \mathrm{E}\left(\phi(\mathbf{w},\mathbf{R})\right)$ is a coherent class of portfolio diversification measure, and a special case of Embrechts et al.'s (2009) class of measures induced by the expected shortfall when F_X is a continuous distribution.

Is the representation form in (31) unique? The following example provides evidence that it is not. It presents a diversification measure that satisfies our axioms, but does not have the representation form (31).

Example 6 (Embrechts et al.'s (2009) Class of Diversification Measures). Consider $\Phi(\mathbf{w}|\mathbf{R})$ such that

$$\Phi(\mathbf{w}|\mathbf{R}) = \mathbf{w}^\top \boldsymbol{\sigma} - \sigma(\mathbf{w}). \tag{34}$$

It is straightforward to verify that $\Phi(\mathbf{w}|\mathbf{R})$ in (34) satisfies our axioms, but does not have the representation form (31).

In sum, we have the following representation theorem.

Proposition 5 (Representation Theorem). *If $\Phi(\mathbf{w}|\mathbf{R})$ satisfies our axioms, then $\Phi(\mathbf{w}|\mathbf{R})$ can take the following representation form*

$$\Phi(\mathbf{w}|\mathbf{R}) = \mathrm{E}\left(\phi(\mathbf{w}, \mathbf{R})\right), \tag{35}$$

where $\phi(\mathbf{w}, \mathbf{R})$ satisfies Size Degeneracy, Risk Degeneracy, Reverse Risk Degeneracy, Duplication Invariance, Size Monotonicity, Translation Invariance, Homogeneity and the following additional properties

(i) $\phi(\mathbf{w}, \mathbf{R})$ *is concave in* $\mathbf{w}$ *for each fixed* $\mathbf{R} \in \mathcal{R}^N$*;*
(ii) $\phi(\mathbf{\Pi w}, \mathbf{\Pi R}) = \phi(\mathbf{w}, \mathbf{R})$ *for all permutations* $\mathbf{\Pi}$*;*
(iii) $\phi(\mathbf{w}, \mathbf{R})$ *is Borel-measurable in* $\mathbf{R}$ *for each fixed* $\mathbf{w}$.

7 Concluding Remarks and Future Research

This paper provides an axiomatic foundation of the measurement of *correlation* diversification in a one-period portfolio theory under the assumption that the investor has complete information about the joint distribution of asset returns. We have specified a set of minimum desirable axioms for measures of *correlation* diversification, and named the measures satisfying these axioms coherent diversification measures.

We have shown that these axioms can be rationalized by *(a)* the expected utility theory if and only if one of the following conditions is satisfied: *(i)* risk is small in the sense of Pratt (1964) and absolute risk aversion is constant, or *(ii)* each asset returns distribution belongs to a location-scale family and the certainty equivalent has a particular additive separable form; *(b)* Yaari's (1987) dual theory if and only if its probability distortion function is convex. These results provide the decision-theoretic foundations of our axiomatic system, and consequently strengthen their desirability, reasonableness and relevance.

We have explored whether portfolio diversification measures such as portfolio variance, diversification ratio, Embrechts et al.'s class of diversification measures and Tasche's class of diversification measures, which are used on the marketplace to manage millions of US dollars and are also in use in the academic world, satisfy those axioms. We have shown that *(i)* portfolio variance satisfies our axioms, but under the very restrictive (if not impossible) condition that the assets have identical variance; *(ii)* the diversification ratio satisfies our axioms; *(iii)* Embrechts et al.'s (1999) and Tasche's (2006) classes of diversification measures satisfy our axioms, but under the conditions that the underlying risk measure is convex (or quasi-convex), homogeneous, translation invariant and reverse lower comonotonic additive. These results strengthen both the axioms and such measures as the diversification ratio and Embrechts et al.'s (1999) and Tasche's (2006) classes of diversification measures. However, they reveal the limits of portfolio variance as an adequate measure of diversification in the mean-variance model.

Finally, we have investigated whether or not our axioms have functional representations. We have shown that our axioms imply a family of representations, but this family is not unique.

Our objective is to offer the first step towards a rigorous theory of *correlation* diversification measures. We believe that with our axiomatic system this is the case. A feasible and desirable direction for future research is to investigate what further axioms could be added or relaxed in order to provide a unique family of representations because our axiomatic system does not.

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A Appendix: Proofs

A.1 Proposition 1

Assume that risk is small. According to Pratt (1964), the approximation of the local risk premium of $u(.)$ is $\pi_u(X) \simeq \frac{1}{2}\mathrm{Var}(X)k(1+\mathrm{E}(X))$, where $k(x) = -\frac{u''(x)}{u'(x)}$ is the measure of local risk aversion of $u(.)$ in a small risk scenario. It follows that $\underline{\varrho}_{C_u}(\mathbf{w}|\mathbf{R}) = -\underline{\varrho}_{\pi_u}(\mathbf{w}|\mathbf{R}) = \frac{1}{2}\left(\sum_{i=1}^N w_i\sigma_i^2 k(1+\mu_i) - \sigma^2(\mathbf{w})k(1+\mu(\mathbf{w}))\right)$. Now let us show that $\underline{\varrho}_{C_u}(\mathbf{w}|\mathbf{R})$ satisfies our axioms if and only if $k(x)$ is a constant function.

A.1.1 Sufficiency

Suppose that $k(x) = c$, $c > 0$. Then $\underline{\varrho}_{C_u}(\mathbf{w}|\mathbf{R}) = c\left(\mathbf{w}^\top\boldsymbol{\sigma}^2 - \sigma^2(\mathbf{w})\right)$. Therefore, we consider $\underline{\underline{\varrho}}_{C_u}(\mathbf{w}|\mathbf{R}) = \mathbf{w}^\top\boldsymbol{\sigma}^2 - \sigma^2(\mathbf{w})$ for the proof.

(C)- Since $\sigma^2(\mathbf{w})$ is convex on $\mathbb{W}$, $\underline{\underline{\varrho}}_{C_u}(\mathbf{w}|\mathbf{R})$ is concave on $\mathbb{W}$.

(SD)- It is straightforward to verify that $\underline{\underline{\varrho}}_{C_u}(\boldsymbol{\delta}_i|\mathbf{R}) = \sigma_i^2 - \sigma_i^2 = 0 = \underline{\Phi}$, for each $i \in \mathcal{I}_N$.

(RD)- Since $A_i = A$, $R_i = R$ for each $i \in \mathcal{I}_N$. Then, $\sigma_i = \sigma_j = \underline{\sigma}$ and $\rho_{ij} = 1$ for each $i, j \in \mathcal{I}_N$ with $\underline{\sigma} > 0$. It follows that $\underline{\underline{\varrho}}_{C_u}(\mathbf{w}|\mathbf{R}) = \underline{\sigma}^2 - \underline{\sigma}^2\left(\sum_{i=1}^N w_i\right)^2 = 0 = \underline{\Phi}$.

(RRD)- Since $w_i \geq 0$, for each $i \in \mathcal{I}_N$ and $\underline{\underline{\varrho}}_{C_u}(\mathbf{w}|\mathbf{R}) = \sum_{i=1}^N w_i\|R_i - R(\mathbf{w})\|_2^2$, $\underline{\underline{\varrho}}_{C_u}(\mathbf{w}|\mathbf{R}) = 0 \Leftrightarrow R_i = R(\mathbf{w})$, for each $i \in \mathcal{I}_N$. The result follows.

(DI)- Since $A_{N+1}^+ = A_k$, $A_i^+ = A_i$ for each $i \in \mathcal{I}_N$,

$$\begin{aligned}
\underline{\underline{\varrho}}_{C_u}(\mathbf{w}_{\mathcal{A}^+}|\mathbf{R}_{\mathcal{A}^+}) &= \sum_{i=1}^{N+1} w_{A_i^+}\sigma_{A_i^+}^2 - \sum_{i,j=1}^{N+1} w_{A_i^+}w_{A_j^+}\sigma_{A_i^+}\sigma_{A_j^+} \\
&= \sum_{i\neq k=1}^{N-1} w_{A_i^+}\sigma_{A_i^+}^2 + \left(w_{A_{N+1}^+} + w_{A_k^+}\right)\sigma_{A_k^+}^2 \\
&\quad - \sum_{i,j\neq k=1}^{N-2} w_{A_i^+}w_{A_j^+}\sigma_{A_i^+}\sigma_{A_j^+} - \sum_{i=1}^{N-2} w_{A_i^+}\left(w_{A_k^+} + w_{A_{N+1}^+}\right)\sigma_{A_i^+}\sigma_{A_k^+}.
\end{aligned}$$

Let $\mathbf{w}_{\mathcal{A}}^{**} = \left(w_{A_1^+}^*, \dots, w_{A_{k-1}^+}^*, w_{A_k^+}^* + w_{A_{N+1}^+}^*, w_{A_{k+1}^+}^*, \dots, w_{A_N^+}^*\right)$ and $\mathbf{w}_{\mathcal{A}^+}^{**} = \left(w_{A_1}^*, \dots, w_{A_{k-1}}^*, \frac{w_{A_k}^*}{2}, w_{A_{k+1}}^*, \dots, \frac{w_{A_k}^*}{2}\right)$. It follows that

$$\begin{aligned}
\underline{\underline{\varrho}}_{C_u}(\mathbf{w}_{\mathcal{A}^+}^*|\mathbf{R}_{\mathcal{A}^+}) &= \underline{\underline{\varrho}}_{C_u}(\mathbf{w}_{\mathcal{A}}^{**}|\mathbf{R}_{\mathcal{A}}) \leq \underline{\underline{\varrho}}_{C_u}(\mathbf{w}_{\mathcal{A}}^*|\mathbf{R}_{\mathcal{A}}), \\
\underline{\underline{\varrho}}_{C_u}(\mathbf{w}_{\mathcal{A}}^*|\mathbf{R}_{\mathcal{A}}) &= \underline{\underline{\varrho}}_{C_u}(\mathbf{w}_{\mathcal{A}^+}^{**}|\mathbf{R}_{\mathcal{A}^+}) \leq \underline{\underline{\varrho}}_{C_u}(\mathbf{w}_{\mathcal{A}^+}^*|\mathbf{R}_{\mathcal{A}^+}).
\end{aligned}$$

Then

$$\underline{\underline{\varrho}}_{C_u}\left(\mathbf{w}^*_{\mathcal{A}^+}|\mathbf{R}_{\mathcal{A}^+}\right) = \underline{\underline{\varrho}}_{C_u}\left(\mathbf{w}^*_{\mathcal{A}}|\mathbf{R}_{\mathcal{A}}\right)$$
$$w^*_{A_i} = w^*_{A_i^+}, \text{ for each } i \neq k,\ i \in \mathcal{I}_N$$
$$w^*_{A_k} = w^*_{A_k^+} + w^*_{A^+_{N+1}}.$$

(M)- Consider a portfolio $\mathbf{w}_{\mathcal{A}^{++}} = \left(\mathbf{w}^*_{\mathcal{A}}, 0\right)$. Portfolio $\mathbf{w}_{\mathcal{A}^{++}}$ is an element of $\mathbb{W}^{N+1}_{\mathcal{A}^{++}}$, so $\underline{\underline{\varrho}}_{C_u}\left(\mathbf{w}^*_{\mathcal{A}^{++}}|\mathbf{R}_{\mathcal{A}^{++}}\right) \geq \underline{\underline{\varrho}}_{C_u}\left(\mathbf{w}_{\mathcal{A}^{++}}|\mathbf{R}_{\mathcal{A}^{++}}\right)$. Since $\underline{\underline{\varrho}}_{C_u}\left(\mathbf{w}_{\mathcal{A}^{++}}|\mathbf{R}_{\mathcal{A}^{++}}\right) = \underline{\underline{\varrho}}_{C_u}\left(\mathbf{w}^*_{\mathcal{A}}|\mathbf{R}_{\mathcal{A}}\right)$, $\underline{\underline{\varrho}}_{C_u}\left(\mathbf{w}^*_{\mathcal{A}^{++}}|\mathbf{R}_{\mathcal{A}^{++}}\right) \geq \underline{\underline{\varrho}}_{C_u}\left(\mathbf{w}^*_{\mathcal{A}}|\mathbf{R}_{\mathcal{A}}\right)$.

(TI,H)- Because covariance is translation invariant and homogeneous of degree two.

(S)- Since $\sigma_i = \sigma_j = \underline{\sigma}$ and $\rho_{ij} = \underline{\rho}$ for each $i, j \in \mathcal{I}_N$ when $R_1, ..., R_N$ is exchangeable, $\underline{\underline{\varrho}}_{C_u}(\mathbf{w}|\mathbf{R}) = \underline{\sigma}^2 - \underline{\sigma}^2\left(\sum_{i=1}^N w_i^2 + \underline{\rho}\sum_{i,j=1}^N w_i w_j\right)$. It is straightforward to verify that $\underline{\underline{\varrho}}_{C_u}(\mathbf{w}|\mathbf{R})$ is symmetric.

A.1.2 Necessity

For the converse, suppose that $\underline{\varrho}_{C_u}(\mathbf{w}|\mathbf{R})$ satisfies our axioms and show that $k(x)$ is a constant function. To do so, we proceed by contradiction. Suppose that $k(x)$ is not a constant function. It is straightforward to verify that $\underline{\varrho}_{C_u}(\mathbf{w}|\mathbf{R})$ satisfies translation invariance and homogeneity if and only if $k(1+x)$ is translation invariant and homogeneous, which is the case only if $k(x)$ is a constant function. This contradicts our hypothesis that $k(x)$ is constant. As a consequence, $\underline{\varrho}_{C_u}(\mathbf{w}|\mathbf{R})$ satisfies our axioms, which implies that $k(x) = c$, $c > 0$.

A.2 Proposition 2

A.2.1 Sufficiency

Follow from the proof of the sufficiency part of Proposition 1.

A.2.2 Necessity

Since asset i returns distributions belong to the location-scale family, form Meyer et al. (1987), $C_u(R) = u^{-1}(U(\mu,\sigma))$, where $\mathrm{E}_u(R) = U(\mu,\sigma) = \int u(\mu + \sigma x)dx$ and $u^{-1}(.)$ is the inverse of $u(.)$. It is obvious that if $\underline{\varrho}_{C_u}(\mathbf{w}|\mathbf{R})$ satisfies our axioms, then $C_u(R) = u^{-1}(U(\mu,\sigma)) = \mu - g(\sigma)$ with $g(.)$ is a strictly increasing, continuous and homogeneous function on $\mathbb{R}_+$.

A.3 Proposition 3

We focus only on $\underline{\varrho}_{C_{\overline{h}}}(\mathbf{w}|\mathbf{R})$.

A.3.1 Sufficiency

Suppose that $\overline{h}(.)$ is convex and let us show that $\underline{\varrho}_{C_{\overline{h}}}(\mathbf{w}|\mathbf{R})$ satisfies our axioms.

(C)- Since $\overline{h}(.)$ is convex, $C_{\overline{h}}(.)$ is convex on $\mathcal{R}$ (Tsanakas and Desli, 2003). It follows that $C_{\overline{h}}(\mathbf{w}|\mathbf{R})$ is convex on $\mathbb{W}$ and consequently, $\underline{\varrho}_{C_{\overline{h}}}(\mathbf{w}|\mathbf{R})$ is concave.

(SD)- Let $\boldsymbol{\delta}_i \in \mathbb{W}$ be a single-asset i portfolio. It is straightforward to show that $\underline{\varrho}_{C_{\overline{h}}}(\boldsymbol{\delta}_i|\mathbf{R}) = C_{\overline{h}}(R_i) - C_{\overline{h}}(R_i) = 0 = \underline{\Phi}$.

(RD)- Since $A_i = A$, $R_i = R$ for each $i \in \mathcal{I}_N$. Then, $\underline{\varrho}_{C_{\overline{h}}}(R_i) = \underline{\varrho}_{C_{\overline{h}}}(R_j)$ for each $i, j \in \mathcal{I}_N$. It follows that $\underline{\varrho}_{C_h}(\mathbf{w}|\mathbf{R}) = C_{\overline{h}}(R) - C_{\overline{h}}(R) = 0 = \underline{\Phi}$.

(RRD)- Since $C_{\overline{h}}(R)$ is coherent, comonotonic additive and non-independent additive, $\underline{\varrho}_{C_{\overline{h}}}(\mathbf{w}|\mathbf{R})$ satisfies Reverse Risk Degeneracy.

(DI)- Follows the proof of Proposition 1.

(M)- Follows the proof of Proposition 1.

(TI,H)- Since $\overline{h}(.)$ is convex, $C_{\overline{h}}(R)$ is translation invariant and homogeneous of degree one. Therefore $\underline{\varrho}_{C_{\overline{h}}}(\mathbf{w}|\mathbf{R})$ is translation invariant and homogeneous of degree one.

(S)- Suppose that $R_1, ..., R_N$ is exchangeable. It is straightforward to verify that $\underline{\varrho}_{C_{\overline{h}}}(\mathbf{w}|\mathbf{R})$ is symmetric.

A.3.2 Necessity

For the converse, suppose that $\underline{\varrho}_{C_{\overline{h}}}(\mathbf{w}|\mathbf{R})$ satisfies our axioms and let us show that $\overline{h}(.)$ is convex. To do so, we proceed by contradiction. Suppose that $h(.)$ is not convex. It is straightforward to verify that $\underline{\varrho}_{C_{\overline{h}}}(\mathbf{w}|\mathbf{R})$ is not concave (Wang et al., 1997).

A.4 Proposition 4

A.4.1 Portfolio variance

A.4.1.1 Sufficiency Suppose that assets have identical variances and show that the portfolio variance satisfies our axioms. It is straightforward to verify that if assets have identical variances i.e $\sigma_i^2 = \underline{\sigma}^2$, then

$$\mathbf{w}^\top \boldsymbol{\sigma}^2 - \sigma^2(\mathbf{w}|\mathbf{R}) = \underline{\sigma}^2 - \sigma^2(\mathbf{w}|\mathbf{R}). \tag{36}$$

From (36) and Proposition 1, it follows that $\sigma^2(\mathbf{w}|\mathbf{R})$ satisfies our axioms.

A.4.1.2 Necessity

For the converse, suppose that $\sigma^2(\mathbf{w}|\mathbf{R})$ our axioms and show that assets have identical variances. To do so, we proceed by contradiction. Suppose that asset variances are not identical and without the loss of generality that $N = 2$ such that $\sigma_1^2 < \sigma_2^2$. Then $\sigma^2(\boldsymbol{\delta}_1|\mathbf{R}) < \sigma^2(\boldsymbol{\delta}_2|\mathbf{R})$. Thus $\sigma^2(\mathbf{w}|\mathbf{R})$ fails Size Degeneracy. From the failure of Size Degeneracy, it is straightforward to prove that $\sigma^2(\mathbf{w}|\mathbf{R})$ also fails Risk Degeneracy and Reverse Risk Degeneracy. This contradicts our hypothesis that $\sigma^2(\mathbf{w}|\mathbf{R})$ satisfies our axioms. As a consequence, if $\sigma^2(\mathbf{w}|\mathbf{R})$ satisfies our axioms, then assets have identical variances.

A.4.2 Diversification ratio

Because the standard-deviation is convex, positive homogeneous (with $\kappa = 1$), translation invariant (with $\eta = 0$) and is reverse lower comonotonic additive, from part (iv) of Proposition 4, DR_σ satisfies our axioms. It follows that $\mathrm{DR}(\mathbf{w}|\mathbf{R}) = \frac{1}{\mathrm{DR}_\sigma}(\mathbf{w}|\mathbf{R})$ also satisfies our axioms.

A.4.3 Embrechts et al.'s (2009) class measures

See the proof of Proposition 3.

A.4.4 Tasche's (2007) class measures

(QC)- Since $\varrho(.)$ is convex and $\sum_{i=1}^{N} w_i \varrho(R_i)$ is linear on $\mathbb{W}$, from Avriel et al. (2010), $\mathrm{DR}_\varrho(\mathbf{w}|\mathbf{R})$ is quasi-concave.

(SD)- $\mathrm{DR}_\varrho(\boldsymbol{\delta}_i|\mathbf{R}) = \frac{\varrho(R_i)}{\varrho(R_i)} = 1$, for each $i \in \mathcal{I}_N$.

(RD)- Since $A_i = A$, $R_i = R$ for each $i \in \mathcal{I}_N$. Then $\mathrm{DR}_\varrho(\boldsymbol{\delta}_i|\mathbf{R}) = \frac{\varrho(R)}{\varrho(R)} = 1 = \underline{\Phi}$ for each $i \in \mathcal{I}_N$.

(RRD)- By assumption that $\varrho(.)$ is reverse upper comonotonic additive.

(DI)- Follows the proof of Proposition 1.

(M)- Follows the proof of Proposition 1.

(TI)- Since $\varrho(.)$ is translation invariant i.e. $\varrho(R + a) = \varrho(R) - \eta a$ with R is a random variable,

$$\mathrm{DR}_\varrho(\mathbf{w}|\mathbf{R}_{\mathcal{A}+a}) = \frac{\varrho(\mathbf{w}^\top \mathbf{R}_{\mathcal{A}}) - \eta a}{\mathbf{w}^\top \varrho(\mathbf{R}_{\mathcal{A}}) - \eta a}.$$

If $\eta = 0$,

$$\mathrm{DR}_\varrho(\mathbf{w}|\mathbf{R}_{\mathcal{A}+a}) = \mathrm{DR}_\varrho(\mathbf{w}|\mathbf{R}_{\mathcal{A}}).$$

If $\eta \neq 0$,

$$\begin{aligned}
\frac{\partial \mathrm{DR}_{\varrho}\big(\mathbf{w}|\mathbf{R}_{\mathcal{A}+a}\big)}{\partial a} = \frac{\eta\left(\varrho\left(\mathbf{w}^{\top}\mathbf{R}_{\mathcal{A}}\right) - \mathbf{w}^{\top}\varrho\left(\mathbf{R}_{\mathcal{A}}\right)\right)}{\left(\mathbf{w}^{\top}\varrho\left(\mathbf{R}_{\mathcal{A}}\right) - \eta\, a\right)^{2}} &\leq 0,\\
\lim_{a\to-\infty} \mathrm{DR}_{\varrho}\big(\mathbf{w}|\mathbf{R}_{\mathcal{A}+a}\big) &= 1,\\
\lim_{a\to+\infty} \mathrm{DR}_{\varrho}\big(\mathbf{w}|\mathbf{R}_{\mathcal{A}+a}\big) &= 1,\\
\lim_{\substack{a\to\frac{\varrho\left(\mathbf{w}^{\top}\mathbf{R}_{\mathcal{A}}\right)}{\eta}\\ a>\frac{\varrho\left(\mathbf{w}^{\top}\mathbf{R}_{\mathcal{A}}\right)}{\eta}}} \mathrm{DR}_{\varrho}\big(\mathbf{w}|\mathbf{R}_{\mathcal{A}+a}\big) &= -\infty,\\
\lim_{\substack{a\to\frac{\varrho\left(\mathbf{w}^{\top}\mathbf{R}_{\mathcal{A}}\right)}{\eta}\\ a<\frac{\varrho\left(\mathbf{w}^{\top}\mathbf{R}_{\mathcal{A}}\right)}{\eta}}} \mathrm{DR}_{\varrho}\big(\mathbf{w}|\mathbf{R}_{\mathcal{A}+a}\big) &= \infty.
\end{aligned}$$

(H)- If $\varrho(.)$ is homogeneous i.e. $\varrho(b\,R) = b^{\kappa}\varrho(R)$ with R is a random variable,

$$\begin{aligned}
\mathrm{DR}_{\varrho}\left(\mathbf{w}|b\,\mathbf{R}\right) &= \frac{\varrho\left(b\,\mathbf{w}^{\top}\mathbf{R}\right)}{\mathbf{w}^{\top}\varrho\left(b\,\mathbf{R}\right)},\\
&= \frac{b^{\kappa}\,\varrho\left(\mathbf{w}^{\top}\mathbf{R}\right)}{b^{\kappa}\,\mathbf{w}^{\top}\varrho\left(\mathbf{R}\right)},\\
\mathrm{DR}_{\varrho}\left(\mathbf{w}|b\,\mathbf{R}\right) &= \mathrm{DR}_{\varrho}\left(\mathbf{w}|\mathbf{R}\right).
\end{aligned}$$

(S)- Follows the proof of Proposition 3.

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