The Principal-Agent Relationship: Two Distributions Satisfying MLRP and CDFC

by Sandrine Spaeter

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Sandrine Spaeter

Sandrine Spaeter is post-doctoral fellow at the Risk Management Chair, École des HEC, and at CRT, Université de Montréal.

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Résumé L'approche au premier ordre, qui consiste à remplacer la contrainte d'incitation de l'agent à produire un effort adéquat par sa condition de premier ordre, est très souvent utilisée dans les problèmes d'agence pour lesquels le principal ne peut observer le niveau d'effort de l'agent. Cette substitution ne remet pas en cause les conditions suffisantes de l'optimisation si la distribution des revenus, ou de la perte, satisfait la propriété de monotonicité du ratio de vraisemblance ainsi que la condition de convexité dans l'effort. Malheureusement, très peu de distributions des revenus satisfont à la fois ces deux propriétés. Nous en proposons deux exemples dans cette note et nous donnons leur équivalent en termes de pertes.

Mots clés : principal-agent, risque moral, distribution, ratio de vraisemblance monotone, convexité dans l'effort.

Abstract The first-order approach, which consists in replacing the incentive compatible constraint by the agent's first order condition, is widely used in agency problems where the principal cannot observe the level of effort chosen by the agent. This substitution is valid with the Monotone Likelihood Ratio Property and the Convex Distribution Function Condition. Unfortunately, revenue distributions seldom present both properties. In this note, we provide two examples of revenue distributions that satisfy MLRP and CDFC. We also give their counterpart in terms of loss distributions.

Keywords: Principal-agent, moral hazard, distribution, monotone likelihood ratio, convexity in effort.

1 Introduction

One main feature in agency problems is that the principal and the agent do not have the same information about the action chosen by the agent. If one thinks about profit maximization, the level of effort that maximizes the expected profit of the principal may be different from the one adopted by the agent. If effort is not verifiable by the principal it is not contractible[1]. Accordingly, the principal has to take into account the fact that the agent privately chooses the action that maximizes her own expected revenue (or utility). This can be done by letting one constraint of his optimization program be the first order condition of the agent's one. This so-called first-order approach is widely used in agency problems despite the fact that it is not always valid. Indeed, the first-order condition of the agent refers to stationary points that may be local minima, saddle-points or local but not global maxima if one does not check that the sufficient conditions are satisfied. Because of the convenience of the first-order approach, its validity is often assumed *ad hoc.* Mirrlees (1975) and also Rogerson (1985) show that it is valid when the revenue distribution satisfies the Monotone Likelihood Ratio Property (MLRP) and the Convex Distribution Function Condition (CDFC). MLRP states that the likelihood ratio is non-decreasing in output, while CDFC deals with the convexity of the distribution in effort. In a more intuitive way, MLRP permits to get a positive relationship between effort and expected gross revenues. With CDFC, it also permits to derive a positive relationship between the observed output and the agent's payoff. But, as mentioned by Jewitt (1988), only a few distributions display both properties.

In this note we build two examples of revenue distributions that display MLRP and CDFC. We also focus on loss distributions in accordance with MLRP and CDFC, which can be used to illustrate many problems where effort affects losses or risks of accident.

Section 2 briefly recalls the first-order approach. Section 3 provides two distributions of gross revenues consistent with MLRP and CDFC and also their counterpart in terms of losses. Section 4 concludes this note.

[1] If the principal can infer some information about the agent's effort, thanks for instance to an imperfect signal, the contract may depend on that signal if it is a sufficient statistic of the level of effort (Holmström, 1979; Jewitt, 1988). It is worth noticing that the level of effort is not a random variable and using the term "sufficient statistic" only drives the idea that the signal carries valuable information about effort.

2 The first-order approach

Consider a principal-agent relationship where the agent privately chooses a level of effort to perform a task delegated by the principal. This effort coupled with Nature yields some random gross revenues that have to be optimally shared between both participants. Effort affects output[2] in the sense of the first order stochastic dominance. The principal's objective is to maximize his expected utility (1) knowing that the payoff to the agent must give her[3] sufficient utility to participate (participation constraint (2.i)) and to adopt the adequate action (incentive constraint (2.ii)).

In this note we focus on the case where no information on action is available to the principal.

The optimization program is as follows:

$$\max_{w(\pi),a} E_F\left[V(\tilde{\pi} - w(\tilde{\pi}))\right] \tag{1}$$

$$s.t.$$

$$\begin{cases} (i) & E_F\left[U(w(\tilde{\pi}))\right] - a \geq U_0 \\ (ii) & a \in \arg\max_{a \in [\underline{a},\overline{a}]} \left(E_F\left[U(w(\tilde{\pi}))\right] - a\right) \end{cases} \tag{2}$$

With:
$\tilde{\pi}$ the gross revenue: $\pi \in [\underline{\pi}, \overline{\pi}]$, $\underline{\pi} \geq 0$.
$F(\pi/a)$ the distribution of $\tilde{\pi}$.
a the level of effort: $a \in [\underline{a}, \overline{a}]$, $\underline{a} \geq 0$.
$U(.)$ the agent's VNM utility : $U'(.) > 0, U''(.) < 0$.
$V(.)$ the principal's VNM utility : $V'(.) > 0, V''(.) \leq 0$.
E the expectation operator.

The cost of effort is identified to the level a chosen by the agent. Assume that $F(\pi/a)$ is a distribution twice continuously differentiable in its two arguments and that its density is well-defined. Still assume a solution to (1)-(2.i)-(2.ii) exists and is differentiable[4]. When a is a continuous variable, the main problem in Program (1)-(2.i)-(2.ii) is that constraint (2.ii) is not tractable such as it stands. An alternative

[2] We will interchangeably use the expressions output and (gross) revenue.
[3] We use the masculine to denote the principal and the feminine to denote the agent.
[4] For a discussion of these statements, the reader is referred to Mirrlees (1974).

method, called the first-order approach, consists in replacing constraint (2.ii) by the first order condition of the agent's optimization program, namely by:

$$\int_{\underline{\pi}}^{\overline{\pi}} U(w(\pi)) f_a(\pi/a) d\pi = 1 \tag{3}$$

Program (1)-(2.i)-(3) can be directly solved. Denote its solution $(a^{**}, w^{**}(\pi))$. Without specific conditions on the revenue distribution, a^{**} may not coincide with solution a^* of (1)-(2.i)-(2.ii), neither does w^{**} with w^*. Indeed, Equation (3) may yield local minima, saddle-points or local but not global maxima, while the initial program implies that the optimal sharing rule $w^*(\pi)$ must be such that the effort level requested by the principal be the one that maximizes the agent's expected utility. Hence Problem (1)-(2.i)-(3) (called the *relaxed Pareto-optimization program* by Rogerson, 1985) is different from (1)-(2.i)-(2.ii) without additional hypotheses. Actually, the equivalence between both programs holds if the distribution of revenues conditional on effort satisfies the Monotone Likelihood Ratio Property (MLRP) and also the Convex Distribution Function Condition (CDFC).[5] Since the settlement of these results, the first-order approach has been widely used in many agency problems.

MLRP states that the likelihood ratio $f_a(\pi/a)/f(\pi/a)$ must be non-decreasing in output π: it is more likely to observe large revenues for a high level of effort. This property implies the first order stochastic dominance ($F_a(\pi/a) < 0$). (But the reverse is not true). Statistics books display several kinds of density functions satisfying MLRP; for instance the normal, the exponential, the Poisson, etc. (with the required mean).

CDFC implies the convexity of $F(\pi/a)$ in a: effort improves the distribution but at a decreasing rate. Unfortunately very few distributions satisfying MLRP also present a convex curve in effort. To our knowledge, the economic literature displays only one example, due to Rogerson (1985):

$$F(\pi/a) = \left(\frac{\pi}{\overline{\pi}}\right)^{a-\underline{a}} \tag{4}$$

Let us notice that this distribution has no density function. Indeed, if the support of the revenues is a strictly positive interval $[\underline{\pi}, \overline{\pi}]$ then F is defined for any a and for any π and it displays a mass point at $\underline{\pi}$: $F(\underline{\pi}/a) > 0$. On the other hand, if one works

[5]Mirrlees (1975) was the first having pointed out this result. See also Grossman and Hart (1983), and Rogerson (1985).

on $[0, \overline{\pi}]$ then $F(0/a)$ equals zero for any $a > \underline{a}$. But its derivative with respect to π evaluated at $\pi = 0$ is no more defined. This specificity of F does not invalidate Rogerson's approach since the author works with discrete variables.

Section 3 presents two continuous revenue distributions that meet both MLRP and CDFC, and for which a density function exists. We also present their counterpart in terms of losses. In the framework of the first-order approach, the latter distributions conditional on action must provide a non-increasing likelihood ratio and must be concave in effort.

3 Two examples of distribution functions

Assume gross revenues are randomly distributed over $[\underline{\pi}, \overline{\pi}]$. $\underline{\pi}$ is zero in Example 1.a and may be positive in Example 2.a. The level of effort a takes continuous values in $[\underline{a}, \overline{a}]$ with $\underline{a} \geq 0$.

Example 1.a

Consider the following function:

$$F^1(\pi/a) = \left[\frac{1}{(a+1)\overline{\pi}}\left(\overline{\pi} - \pi\right) + 1\right].\frac{\pi}{\overline{\pi}} \tag{5}$$

This function is twice continuously differentiable over $[0, \overline{\pi}]$ and over $[\underline{a}, \overline{a}]$. It is equal to zero at $\pi = 0$, strictly positive for any value in $]0, \overline{\pi}[$, and equal to one at $\pi = \overline{\pi}$, whatever the value of a. Also it is increasing in π:

$$\frac{d}{d\pi}F^1(\pi/a) = f^1(\pi/a) = \left[\frac{1}{(a+1)\overline{\pi}}\left(\overline{\pi} - 2\pi\right) + 1\right].\frac{1}{\overline{\pi}} \geq 0 \tag{6}$$

Due to these properties, F^1 is a distribution of π conditional on the effort level a and f^1 is the associated density. Let us look at the first and second derivatives of F^1 with respect to a. We have

$$F^1_a(\pi/a) = \frac{-\pi}{(a+1)^2\overline{\pi}^2}\left(\overline{\pi} - \pi\right),$$

with

$$\begin{cases} (i) \quad F^1_a(\pi/a) < 0, \ \forall \pi \in]0, \overline{\pi}[\\ (ii) \quad F^1_a(0/a) = F_a(\overline{\pi}/a) = 0 \end{cases} \tag{7}$$

Property (7.i) refers to the first order stochastic dominance, which is implied by MLRP. But since the reverse is not true we still have to show that F^1 does satisfy MLRP. By calculating the likelihood ratio $f_a^1(\pi/a)/f^1(\pi/a)$ (for $a > 0$) and by differentiating it with respect to π, one can show that it is increasing in π, so that F^1 satisfies MLRP. This is demonstrated in section A of Appendix 1. Function (5) also satisfies CDFC. Indeed:

$$F_{aa}^1(\pi/a) = \frac{2\pi}{(a+1)^3.\overline{\pi}^2}(\overline{\pi} - \pi) > 0, \ \forall a, \ \forall \pi \in]0, \overline{\pi}[\tag{8}$$

Hence F^1 satisfies MLRP and CDFC. ■

Density f^1 is depicted on Figure 1 with respect to the gross revenue, which variate in $[0, 100]$, and for two different levels of effort. It is decreasing in π. Distribution F^1 is related to gross revenues and, consequently, to a problem where the objective of the principal is to maximize profits through a task performed by an agent. But some agency problems also deal with damages, such as in insurance for instance. The agent (the insured person) may influence her conditional accident cost by choosing some specific action like driving carefully (or fast), carrying (or not) her seat belt, etc. If the insurer cannot observe the level or the type of action chosen by the insured when fixing the price of insurance, one has to cope with *ex ante* moral hazard (Winter, 1992). In such a context MLRP says that the higher the level of effort, the higher the likelihood of observing a damage with not too large severity. As a direct consequence of MLRP, the loss distribution increases in effort. Furthermore, when the insurer wants to maximize his profits under participation and incentive constraints the sufficient condition for a global maximum now refers to the concavity of the loss distribution with respect to a. In summary, if we denote $\tilde{l}$ the risk of damage with $l \in \left[\underline{l}, \overline{l}\right]$, $\underline{l} \geq 0$, and a the effort with $a \in [\underline{a}, \overline{a}]$, we should have:

$$\begin{cases} (i) & \frac{\partial}{\partial l}\left(\frac{f_a(l/a)}{f(l/a)}\right) \leq 0 \\ (ii) & F_{aa}(l/a) < 0 \ \forall l \in \left]\underline{l}, \overline{l}\right[\end{cases} \tag{9}$$

Condition (9.i) implies $F_a(l/a) > 0$. The following function, which is the counterpart of Example 1.a, displays both properties:

Example 1.b

$$\begin{array}{l} F^2(l/a) = \left[\frac{(a+1)^{1/2}}{k}(\bar{l} - l) + 1\right] \cdot \frac{l}{\bar{l}}, \\ with \quad k > \bar{l}(\bar{a}+1)^{1/2} \quad and \quad l \in \left[0, \bar{l}\right] \end{array} \tag{10}$$

■

The condition imposed on the scalar k ensures the strict positivity of the density function. The properties of F^2 are given in section B of Appendix 1. Density f^2 is depicted on Figure 2 with respect to l, which varies in $[0, 150]$, and for two different levels of effort. It is decreasing in loss.

An other environment where the properties of F^2 are welcome deals with the bank/entrepreneur relationship. Assume the firm needs some external funds in order to start a risky project. She can take some actions to prevent an accident within the production process for instance. The *ex post* profits depend on the occurrence of a damage and, consequently, the chances for the bank to be reimbursed are affected by the level of prevention adopted by the firm. Still here, the distribution of losses should satisfy Properties (9).

Hereafter, we provide a second example of revenue distribution.

Example 2.a

Consider the following function:

$$G^1(\pi/a) = (a+k)^{(\pi - \overline{\pi})}.\left(\frac{\pi - \underline{\pi}}{\overline{\pi} - \underline{\pi}}\right);\ k > 1 \tag{11}$$

Here, the lower bound $\underline{\pi}$ may be positive or zero. $G^1(\pi/a)$ is twice continuously differentiable over $[\underline{\pi}, \overline{\pi}]$ and over $[\underline{a}, \overline{a}]$. It displays the following properties:

$$\begin{cases} G^1(\pi/a) > 0,\ \forall a,\ \forall \pi > \underline{\pi} \\ G^1(\underline{\pi}/a) = 0,\ G^1(\overline{\pi}/a) = 1 \end{cases}$$

With $k > 1$ it is also strictly increasing in π whatever the value of a:

$$\frac{\partial G^1(\pi/a)}{\partial \pi} = g^1(\pi/a) = \frac{(a+k)^{(\pi - \overline{\pi})}}{(\overline{\pi} - \underline{\pi})}\left[1 + \ln(a+k).(\pi - \underline{\pi})\right] > 0 \tag{12}$$

Thus G^1 displays the properties of a distribution function and g^1 is the associated density function.

The first derivative of G^1 with respect to a is $G^1_a(\pi/a) = (\pi-\overline{\pi})(a+k)^{(\pi-\overline{\pi}-1)}.\left(\frac{\pi-\underline{\pi}}{\overline{\pi}-\underline{\pi}}\right)$. So that:

$$\begin{cases} G^1_a(\pi/a) < 0, \ \forall \pi \in \left]\underline{\pi}, \overline{\pi}\right[\\ G^1_a(\underline{\pi}/a) = G^1_a(\overline{\pi}/a) = 0 \end{cases}$$

Moreover:

$$G^1_{aa}(\pi/a) = (\pi - \overline{\pi} - 1)(\pi - \overline{\pi})(a + k)^{(\pi-\overline{\pi}-2)}.\left(\frac{\pi - \underline{\pi}}{\overline{\pi} - \underline{\pi}}\right) > 0, \ \forall \pi \in \left]\underline{\pi}, \overline{\pi}\right[$$

Distribution G^1 satisfies the first order stochastic dominance and also CDFC. Now, we have to show that it still satisfies MLRP. This is demonstrated in section C of Appendix 1.

Finally, Function G^1 satisfies both MLRP and CDFC. ■

It is worth noticing that g^1 increases in output, whereas f^1 decreases. Density g^1 is depicted on Figure 3 with $\pi \in [0, 10]$.

Hereafter, we give the counterpart of G^1 in terms of losses. Distribution G^2 is defined for levels of effort between $[\underline{a}, 1]$, $\underline{a} \geq 0$. [6] Properties and computations are given in section D of Appendix 1.

Example 2.b

For $l \in \left[\underline{l}, \overline{l}\right]$, with $\underline{l} \geq 0$, and $a \in [\underline{a}, 1]$, with $\underline{a} \geq 0$:

$$G^2(l/a) = (a + 1)^{\left(\frac{\overline{l}-l}{\overline{l}-\underline{l}}\right)}.\left(\frac{l - \underline{l}}{\overline{l} - \underline{l}}\right)$$

■

Since $\underline{l}$ may be positive or zero, the distribution G^2 can be used for risks of damage having a continuous distribution over all states of nature (the no-accident state included) or for situations where the no-accident state presents a mass point, while positive damages have a continuous distribution. Density g^2 is depicted on Figure 4 with respect to l, which takes values in $[0, 10]$, and for two different levels of effort.

[6] This hypothesis is made in order to simplify the calculus already complicated with G^2. The generalization to any positive interval for the level of effort states as:

$$G^3(l/a) = (a + 1)^{\left(\frac{\overline{l}-l}{\overline{l}-\underline{l}}\right)^k}.\left(\frac{l - \underline{l}}{\overline{l} - \underline{l}}\right), \quad with \quad k > 0$$

Before concluding, we give the means and variances for each example of distribution in Table 1. They are calculated in Appendix 2, with $\pi \in [0, \overline{\pi}]$ and $l \in \left[0, \overline{l}\right]$.

Table 1

	Mean	**Variance**
$F^1(\pi/a)$	$\frac{\overline{\pi}}{2}\left(1 - \frac{1}{3(a+1)}\right)$	$\frac{\overline{\pi}^3}{12}\left(4 + b(3b-6)\right)$
$G^1(\pi/a)$	$\overline{\pi} + \frac{1}{\ln(a+k)}\left(\frac{1}{\overline{\pi}\ln(a+k)}\left(1 - (a+k)^{(-\overline{\pi})}\right) - 1\right)$	$\frac{\overline{\pi}^3}{3} + \overline{\pi}c(\overline{\pi} + c)$
$F^2(l/a)$	$\frac{\overline{l}}{2}\left(1 - \frac{\overline{l}(a+1)^{1/2}}{3k}\right)$	$\frac{\overline{l}^3}{3}\left(1 - h + \frac{h^2}{3}\right)$
$G^2(l/a)$	$\overline{l}\left(1 + \frac{1}{\ln(a+1)}\left(1 - \frac{a}{\ln(a+1)}\right)\right)$	$\overline{l}^3\left(\frac{1}{3} + m + m^2\right)$

$$\text{with} \begin{cases} b = \left(1 - \frac{1}{3(a+1)}\right) \\ c = \frac{1}{\ln(a+k)}\left(\frac{1}{\overline{\pi}\ln(a+k)}\left(1 - (a+k)^{(-\overline{\pi})}\right) - 1\right) \\ h = \frac{1}{2}\left(3 - \frac{\overline{l}(a+1)^{1/2}}{k}\right) \\ m = \frac{1}{\ln(a+1)}\left(1 - \frac{a}{\ln(a+1)}\right) \quad \textit{and} \quad a \in \left]0, 1\right] \end{cases}$$

4 Conclusion

We have provided two examples of revenue distributions that satisfy MLRP and CDFC. Accordingly, they can be used to illustrate many agency problems - such as employee/employer relationships, sharecropping, bank/firm relationships or law enforcement[7] - solved with the first-order approach.

We have still given their counterpart in terms of loss distributions, so that insurance problems (Winter, 1992), models with environmental risk (Boyer and Laffont, 1997; Dionne and Spaeter, 1998) and, more generally, models with risks of losses (Brander and Spencer, 1989; Dionne, Gagné, Gagnon and Vanasse, 1997) may also be illustrated thanks to them. Up to now only one distribution, due to Rogerson (1985), was presented in the literature and illustrations were seldom possible notably when the agency

[7] See for instance Harris and Raviv (1978) for formalizations of such problems.

problem deals with two random variables. The examples we just provided here enlarge the set of admissible distributions.

Our distributions can also be applied to problems where a risk-averse agent has to split his initial wealth between a risky asset and a safe one. Indeed, it is known that intuitive comparative static results can be obtained about the behavior of the agent following an increase in risk that satisfies MLRP (Landsberger and Meilijson, 1990; Ormiston and Schlee, 1993). Lastly, the distributions presented in this paper still hold for the MPR (*Monotone Probability Ratio*) Property established by Eeckhoudt and Gollier (1995) and related to the ratio of cumulative distributions rather than to that of density functions.

APPENDIX 1

A. Example 1.a.

For Example 1.a we have:

$$\begin{cases} f^1(\pi/a) = \left[\frac{1}{(a+1)\overline{\pi}}\left(\overline{\pi}-2\pi\right)+1\right].\frac{1}{\overline{\pi}} \\ f^1_a(\pi/a) = \frac{-1}{(a+1)^2.\overline{\pi}^2}\left(\overline{\pi}-2\pi\right) \end{cases}$$

After simplification the likelihood ratio equals:

$$\frac{f^1_a(\pi/a)}{f^1(\pi/a)} = \frac{2\pi-\overline{\pi}}{(a+1)\left(\overline{\pi}-2\pi\right)+(a+1)^2\overline{\pi}}$$

Differentiating it with respect to π and simplifying the result leads to

$$\frac{\partial}{\partial\pi}\left(\frac{f^1_a(\pi/a)}{f^1(\pi/a)}\right) = \frac{2(a+1)^2\overline{\pi}}{\left[(a+1)\left(\overline{\pi}-2\pi\right)+(a+1)^2\overline{\pi}\right]^2},$$

which is always strictly positive (for $a > 0$). Consequently, F^1 satisfies MLRP. ♦

B. Example 1.b.

We have for any $l \in \left[0, \overline{l}\right]$:

$$F^2(l/a) = \left[\frac{(a+1)^{1/2}}{k}(\overline{l}-l)+1\right].\frac{l}{\overline{l}}, \quad with \quad k > \overline{l}(\overline{a}+1)^{1/2}$$

Distribution F^2 satisfies: $F^2(0/a) = 0$ and $F^2(\bar{l}/a) = 1$. Also F^2 is strictly positive for any $l > 0$. The associated density function is:

$$f^2(l/a) = \left[\frac{(a+1)^{1/2}}{k}(\bar{l} - 2l) + 1\right] \cdot \frac{1}{\bar{l}}$$

It is strictly positive for any l and any a if the term into brackets is strictly positive at point $(\bar{l}, \overline{a})$ because f^2 is increasing in a and decreasing in l. This is satisfied for any k strictly larger than $\bar{l}(\overline{a}+1)^{1/2}$.

Concerning F^2 we have:

$$\begin{cases} F_a^2(l/a) = \frac{(a+1)^{-1/2}}{2k}(\bar{l} - l).\frac{l}{\bar{l}} > 0 \quad \forall a, \forall l > 0 \\ F_a^2(0/a) = F_a(\bar{l}/a) = 0 \\ F_{aa}^2(l/a) = -\frac{(a+1)^{-3/2}}{4k}(\bar{l} - l).\frac{l}{\bar{l}} < 0, \quad \forall a, \quad \forall l > 0 \end{cases}$$

Now, let us show that F^2 also satisfies MLRP. We have $f_a^2(l/a) = \frac{(a+1)^{-1/2}}{2k}(\bar{l} - 2l).\frac{1}{\bar{l}}$. And after simplification:

$$\frac{f_a^2(l/a)}{f^2(l/a)} = \frac{(a+1)^{-1}(\bar{l} - 2l)}{2[(\bar{l} - 2l) + k]}$$

Differentiating this likelihood ratio with respect to l leads to:

$$\begin{aligned} \frac{\partial}{\partial l}\left[\frac{f_a^2(l/a)}{f^2(l/a)}\right] &= \frac{-4(a+1)^{-1}\left[(\bar{l} - 2l) + k\right] + 4(a+1)^{-1}(\bar{l} - 2l)}{4\left[(\bar{l} - 2l) + k\right]^2} \\ &= \frac{-(a+1)^{-1}k}{\left[(\bar{l} - 2l) + k\right]^2} < 0 \end{aligned}$$

Then F^2 also satisfies MLRP. ♦

C. Example 2.a.

For Example 2.a we have:

$$\begin{cases} g^1(\pi/a) = \frac{(a+k)^{(\pi - \overline{\pi})}}{(\overline{\pi} - \underline{\pi})}\left[1 + \ln(a+k).(\pi - \underline{\pi})\right] \\ g_a^1(\pi/a) = \frac{(a+k)^{(\pi - \overline{\pi} - 1)}}{(\overline{\pi} - \underline{\pi})}\left[(\pi - \underline{\pi}) + (\pi - \overline{\pi})\left(1 + \ln(a+k)(\pi - \underline{\pi})\right)\right] \end{cases}$$

After simplification the likelihood ratio equals:

$$\frac{g_a^1(\pi/a)}{g^1(\pi/a)} = \frac{(a+k)^{-1}\left[(\pi-\underline{\pi}) + (\pi-\overline{\pi})\left(1+\ln(a+k)(\pi-\underline{\pi})\right)\right]}{(1+\ln(a+k).(\pi-\underline{\pi}))}$$

Differentiating this last ratio with respect to π leads to

$$\begin{aligned}\frac{\partial}{\partial\pi}\left(\frac{g_a^1(\pi/a)}{g^1(\pi/a)}\right) &= \{(2+(2\pi-\underline{\pi}-\overline{\pi})\ln(a+k))\,(1+\ln(a+k).(\pi-\underline{\pi})) \\ &\quad -((\pi-\underline{\pi})+(\pi-\overline{\pi})\,(1+\ln(a+k)(\pi-\underline{\pi})))\ln(a+k)\}\,/\mathcal{D},\end{aligned}$$

with $\mathcal{D} = (a+k)\,(1+\ln(a+k).(\pi-\underline{\pi}))^2$. By developing each term in the right-hand-side and by simplifying we obtain finally

$$\frac{\partial}{\partial\pi}\left(\frac{g_a^1(\pi/a)}{g^1(\pi/a)}\right) = \frac{2\,(1+(\pi-\underline{\pi})\ln(a+k)) + (\pi-\underline{\pi})^2\left[\ln(a+k)\right]^2}{\mathcal{D}},$$

which is strictly positive. As a result, G^1 satisfies MLRP. ♦

D. Example 2.b.

We have $G^2(l/a) = (a+1)^{\left(\frac{\overline{l}-l}{\overline{l}-\underline{l}}\right)}.\left(\frac{l-\underline{l}}{\overline{l}-\underline{l}}\right)$. Assume $l \in [\underline{l}, \overline{l}]$ with $\underline{l} \geq 0$ and $a \in [\underline{a}, 1]$ with $\underline{a} \geq 0$.

In the course we use the following notation: $C(l) = \left(\frac{\overline{l}-l}{\overline{l}-\underline{l}}\right)$ and $D(l) = \left(\frac{l-\underline{l}}{\overline{l}-\underline{l}}\right)$. We have: $0 \leq C(l) \leq 1$ and $0 \leq D(l) \leq 1$. G^2 satisfies the following properties: $G^2(\underline{l}/a) = 0$ and $G^2(\overline{l}/a) = 1$. Also G^2 is strictly positive for any $l > \underline{l}$. The associated density function is:

$$\begin{aligned}g^2(l/a) &= \frac{(a+1)^{C(l)}}{(\overline{l}-\underline{l})} - \frac{1}{(\overline{l}-\underline{l})}\ln(a+1).(a+1)^{C(l)}.D(l) \\ &= \frac{(a+1)^{C(l)}}{(\overline{l}-\underline{l})}\left[1-\ln(a+1).D(l)\right]\end{aligned}$$

This function is strictly positive for any a and any l if the term into brackets is strictly positive. To get this property it is sufficient that this term be positive at point $(\overline{a}, \overline{l})$. With $\overline{a} = 1$ this is always true.

We also have:

$$\begin{cases} G^2_a(l/a) = C(l)(a+1)^{C(l)-1}D(l) > 0, \ \forall \underline{l} < l < \bar{l} \\ G^2_a(\underline{l}/a) = G^2_a(\bar{l}/a) = 0 \\ G^2_{aa}(l/a) = C(l)(C(l)-1)(a+1)^{C(l)-2}D(l) < 0, \ \forall \underline{l} < l < \bar{l} \end{cases}$$

Now, let us show that G^2 also satisfies MLRP. We have:

$$\begin{aligned} g^2_a(l/a) &= C(l)\frac{(a+1)^{C(l)-1}}{(\bar{l}-\underline{l})}\left[1-\ln(a+1).D(l)\right] - \frac{(a+1)^{C(l)}}{(\bar{l}-\underline{l})}\frac{1}{(a+1)}D(l) \\ &= \frac{(a+1)^{C(l)-1}}{(\bar{l}-\underline{l})}\left[(1-\ln(a+1).D(l))C(l) - D(l)\right] \end{aligned}$$

Hence:

$$\frac{g^2_a(l/a)}{g^2(l/a)} = \frac{(a+1)^{-1}\left[(1-\ln(a+1).D(l))C(l)-D(l)\right]}{(1-\ln(a+1).D(l))}$$

Differentiating this likelihood ratio with respect to l leads to:

$$\begin{aligned} \frac{\partial}{\partial l}\left[\frac{g^2_a(l/a)}{g^2(l/a)}\right] &= \{(1-\ln(a+1).D(l))[(1-\ln(a+1).D(l))C'(l) - \ln(a+1)D'(l)C(l) \\ &\quad -D'(l)] + \ln(a+1)D'(l)\left[(1-\ln(a+1).D(l))C(l)-D(l)\right]\} \\ &\quad /(a+1)(1-\ln(a+1).D(l))^2 \end{aligned}$$

Knowing that $C'(l) = -1/(\bar{l}-\underline{l})$ and that $D'(l) = 1/(\bar{l}-\underline{l})$ we obtain after simplifications:

$$\frac{\partial}{\partial l}\left[\frac{g^2_a(l/a)}{g^2(l/a)}\right] = \frac{-1-(1-\ln(a+1).D(l))^2}{(a+1)(\underline{l}-\bar{l})(1-\ln(a+1).D(l))^2} < 0$$

So, G^2 also satisfies MLRP. ⧫

APPENDIX 2

In this appendix, we compute the means and the variances for each example of distribution. Revenues variate in $[0, \overline{\pi}]$ and losses in $\left[0, \overline{l}\right]$.

■ Example 1.a

Distribution F^1 is as follows: $F^1(\pi/a) = \left[\frac{1}{(a+1)\overline{\pi}}(\overline{\pi} - \pi) + 1\right] . \frac{\pi}{\overline{\pi}}$.
Its density function is $f^1(\pi/a) = \left[\frac{1}{(a+1)\overline{\pi}}(\overline{\pi} - 2\pi) + 1\right] . \frac{1}{\overline{\pi}}$.

- **The mean** $E_{F^1}(\widetilde{\pi})$

We have:

$$\begin{aligned} E_{F^1}(\widetilde{\pi}) &= \int_0^{\overline{\pi}} \left(\frac{1}{(a+1)\overline{\pi}^2} \left(\overline{\pi}\pi - 2\pi^2\right) + \frac{\pi}{\overline{\pi}} \right) d\pi \\ &= \left[\frac{1}{(a+1)\overline{\pi}^2} \left(\frac{\overline{\pi}\pi^2}{2} - \frac{2\pi^3}{3} \right) + \frac{\pi^2}{2\overline{\pi}} \right]_0^{\overline{\pi}} \\ &= \frac{-\overline{\pi}^3}{6(a+1)\overline{\pi}^2} + \frac{\overline{\pi}}{2} \end{aligned}$$

And finally $E_{F^1}(\widetilde{\pi}) = \frac{\overline{\pi}}{2}\left(1 - \frac{1}{3(a+1)}\right)$.

- **The variance** $V_{F^1}(\widetilde{\pi})$

We have $V_{F^1}(\widetilde{\pi}) = \int_0^{\overline{\pi}} \left(\pi - \frac{\overline{\pi}}{2}\left(1 - \frac{1}{3(a+1)}\right)\right)^2 d\pi$.
Let us adopt the following notation:

$$b = \left(1 - \frac{1}{3(a+1)}\right) \tag{13}$$

Thus:

$$\begin{aligned}
V_{F^1}(\widetilde{\pi}) &= \int_0^{\overline{\pi}} \left(\pi^2 - \pi\overline{\pi}b + \frac{\overline{\pi}^2 b^2}{4}\right) d\pi \\
&= \left[\frac{\pi^3}{3} - \frac{\pi^2\overline{\pi}b}{2} + \frac{\pi\overline{\pi}^2 b^2}{4}\right]_0^{\overline{\pi}} \\
&= \frac{\overline{\pi}^3}{3} - \frac{\overline{\pi}^3 b}{2} + \frac{\overline{\pi}^3 b^2}{4} \\
&= \overline{\pi}^3 \left(\frac{1}{3} - \frac{b}{2} + \frac{b^2}{4}\right)
\end{aligned}$$

And finally $V_{F^1}(\widetilde{\pi}) = \frac{\overline{\pi}^3}{12}\left(4 + b(3b-6)\right)$, where b is given by (13).

■ Example 1.b

Recall that F^2 is defined as follows

$$F^2(l/a) = \left[\frac{(a+1)^{1/2}}{k}(\overline{l} - l) + 1\right] . \frac{l}{\overline{l}},$$

with $k > \overline{l}(\overline{a}+1)^{1/2}$, and its density function is:

$$f^2(l/a) = \left[\frac{(a+1)^{1/2}}{k}(\overline{l} - 2l) + 1\right] . \frac{1}{\overline{l}}$$

- **The mean** $E_{F^2}(\widetilde{l})$

We have:

$$\begin{aligned}
E_{F^2}(\widetilde{l}) &= \frac{1}{\overline{l}} \int_0^{\overline{l}} \left(\frac{(a+1)^{1/2}}{k}(\overline{l}l - 2l^2) + l\right) dl \\
&= \frac{1}{\overline{l}} \left[\frac{(a+1)^{1/2}}{k}\left(\frac{\overline{l}l^2}{2} - \frac{2l^3}{3}\right) + \frac{l^2}{2}\right]_0^{\overline{l}} \\
&= \frac{-\overline{l}^2(a+1)^{1/2}}{6k} + \frac{\overline{l}}{2}
\end{aligned}$$

And finally $E_{F^2}(\widetilde{l}) = \frac{\overline{l}}{2}\left(1 - \frac{\overline{l}(a+1)^{1/2}}{3k}\right)$, which is positive because of the condition imposed on k.

- **The variance $V_{F^2}(\widetilde{l})$**

We have $V_{F^2}(\widetilde{l}) = \int\limits_0^{\bar{l}} \left(l - \frac{\bar{l}}{2}\left(1 - \frac{\bar{l}(a+1)^{1/2}}{3k}\right)\right)^2 dl.$

Let us adopt the following notation: $e = \frac{\bar{l}}{2}\left(1 - \frac{\bar{l}(a+1)^{1/2}}{3k}\right)$. Thus:

$$\begin{aligned} V_{F^2}(\widetilde{l}) &= \int\limits_0^{\bar{l}} (l^2 - 2le + e^2) dl \\ &= \left[\frac{l^3}{3} - l^2 e + le^2\right]_0^{\bar{l}} \end{aligned}$$

And finally $V_{F^2}(\widetilde{l}) = \frac{\bar{l}^3}{3}\left(1 - h + \frac{h^2}{3}\right)$, with $h = \frac{1}{2}\left(3 - \frac{\bar{l}(a+1)^{1/2}}{k}\right)$.

■ Example 2.a

Distribution G^1 is defined as follows: $G^1(\pi/a) = (a+k)^{(\pi - \overline{\pi})}.\left(\frac{\pi - \underline{\pi}}{\overline{\pi} - \underline{\pi}}\right)$ with $k > 1$.

Its density function is $g^1(\pi/a) = \frac{(a+k)^{(\pi-\overline{\pi})}}{(\overline{\pi} - \underline{\pi})}\left[1 + \ln(a+k).(\pi - \underline{\pi})\right]$.

- **The mean $E_{G^1}(\widetilde{\pi})$**

With $\underline{\pi} = 0$, we have:

$$\begin{aligned} E_{G^1}(\widetilde{\pi}) &= \frac{1}{\overline{\pi}} \int\limits_0^{\overline{\pi}} (a+k)^{(\pi-\overline{\pi})} \left(\pi + \pi^2 \ln(a+k)\right) d\pi \\ &= \frac{1}{\overline{\pi}} \{ \left[(a+k)^{(\pi-\overline{\pi})} \frac{\pi^2}{2}\right]_0^{\overline{\pi}} - \int\limits_0^{\overline{\pi}} \ln(a+k)(a+k)^{(\pi-\overline{\pi})} \frac{\pi^2}{2} d\pi \\ &\quad + \int\limits_0^{\overline{\pi}} \ln(a+k)(a+k)^{(\pi-\overline{\pi})} \pi^2 d\pi \} \\ &= \frac{1}{\overline{\pi}} \left\{ \frac{\overline{\pi}^2}{2} + \int\limits_0^{\overline{\pi}} \ln(a+k)(a+k)^{(\pi-\overline{\pi})} \frac{\pi^2}{2} d\pi \right\} \\ &= \frac{1}{\overline{\pi}} \left\{ \frac{\overline{\pi}^2}{2} + \left[(a+k)^{(\pi-\overline{\pi})} \frac{\pi^2}{2}\right]_0^{\overline{\pi}} - \int\limits_0^{\overline{\pi}} (a+k)^{(\pi-\overline{\pi})} \pi d\pi \right\} \end{aligned}$$

$$= \frac{1}{\overline{\pi}}\left\{\overline{\pi}^2 - \left[\frac{(a+k)^{(\pi-\overline{\pi})}}{\ln(a+k)}\pi\right]_0^{\overline{\pi}} + \int_0^{\overline{\pi}} \frac{(a+k)^{(\pi-\overline{\pi})}}{\ln(a+k)} d\pi\right\}$$

$$= \frac{1}{\overline{\pi}}\left\{\overline{\pi}^2 - \frac{\overline{\pi}}{\ln(a+k)} + \left[\frac{(a+k)^{(\pi-\overline{\pi})}}{(\ln(a+k))^2}\right]_0^{\overline{\pi}}\right\}$$

And finally $E_{G^1}(\widetilde{\pi}) = \overline{\pi} + \frac{1}{\ln(a+k)}\left(\frac{1}{\overline{\pi}\ln(a+k)}\left(1-(a+k)^{(-\overline{\pi})}\right) - 1\right)$.

- **The variance** $V_{G^1}(\widetilde{\pi})$

We have:

$$V_{G^1}(\widetilde{\pi}) = \int_0^{\overline{\pi}} \left(\pi - \overline{\pi} - \frac{1}{\ln(a+k)}\left(\frac{1}{\overline{\pi}\ln(a+k)}\left(1-(a+k)^{(-\overline{\pi})}\right) - 1\right)\right)^2 d\pi$$

Let us adopt the following notation:

$$c = \frac{1}{\ln(a+k)}\left(\frac{1}{\overline{\pi}\ln(a+k)}\left(1-(a+k)^{(-\overline{\pi})}\right) - 1\right) \tag{14}$$

Thus:

$$\begin{aligned} V_{G^1}(\widetilde{\pi}) &= \int_0^{\overline{\pi}} \left((\pi-\overline{\pi})^2 - 2(\pi-\overline{\pi})c + c^2\right) d\pi \\ &= \int_0^{\overline{\pi}} \left(\pi^2 - 2\pi\overline{\pi} + \overline{\pi}^2 - 2(\pi-\overline{\pi})c + c^2\right) d\pi \\ &= \left[\frac{\pi^3}{3} - \pi^2\overline{\pi} + \pi\overline{\pi}^2 - \pi^2 c + 2\pi\overline{\pi}c + \pi c^2\right]_0^{\overline{\pi}} \\ &= \frac{\overline{\pi}^3}{3} + \overline{\pi}^2 c + \overline{\pi}c^2 \end{aligned}$$

And finally $V_{G^1}(\widetilde{\pi}) = \frac{\overline{\pi}^3}{3} + \overline{\pi}c(\overline{\pi}+c)$, with c defined by (14).

■ Example 2.b

For $a \in [\underline{a}, 1]$, with $\underline{a} > 0$, G^2 is defined by:

$$G^2(l/a) = (a+1)^{\left(\frac{\bar{l}-l}{\bar{l}-\underline{l}}\right)} . \left(\frac{l-\underline{l}}{\bar{l}-\underline{l}}\right)$$

Its density function is $g^2(l/a) = \frac{(a+1)^{\left(\frac{\bar{l}-l}{\bar{l}-\underline{l}}\right)}}{(\bar{l}-\underline{l})}\left[1-\ln(a+1).\left(\frac{l-\underline{l}}{\bar{l}-\underline{l}}\right)\right]$.

- **The mean** $E_{G^2}(\widetilde{l})$

We have:

$$\begin{aligned}
E_{G^2}(\widetilde{l}) &= \frac{1}{\bar{l}}\int_0^{\bar{l}}(a+1)^{\left(\frac{\bar{l}-l}{\bar{l}}\right)}\left(l-\ln(a+1)\frac{l^2}{\bar{l}}\right)dl \\
&= \frac{1}{\bar{l}}\{\left[(a+1)^{\left(\frac{\bar{l}-l}{\bar{l}}\right)}\frac{l^2}{2}\right]_0^{\bar{l}} + \int_0^{\bar{l}}(a+1)^{\left(\frac{\bar{l}-l}{\bar{l}}\right)}\ln(a+1)\frac{l^2}{2\bar{l}}dl \\
&\quad - \int_0^{\bar{l}}(a+1)^{\left(\frac{\bar{l}-l}{\bar{l}}\right)}\ln(a+1)\frac{l^2}{\bar{l}}dl\} \\
&= \frac{1}{\bar{l}}\left\{\frac{\bar{l}^2}{2} - \int_0^{\bar{l}}(a+1)^{\left(\frac{\bar{l}-l}{\bar{l}}\right)}\ln(a+1)\frac{l^2}{2\bar{l}}dl\right\} \\
&= \frac{1}{\bar{l}}\left\{\frac{\bar{l}^2}{2} - \left[-(a+1)^{\left(\frac{\bar{l}-l}{\bar{l}}\right)}\frac{l^2}{2}\right]_0^{\bar{l}} - \int_0^{\bar{l}}(a+1)^{\left(\frac{\bar{l}-l}{\bar{l}}\right)}l\,dl\right\} \\
&= \frac{1}{\bar{l}}\left\{\bar{l}^2 - \left[\frac{-(a+1)^{\left(\frac{\bar{l}-l}{\bar{l}}\right)}}{\ln(a+1)}\bar{l}l\right]_0^{\bar{l}} - \int_0^{\bar{l}}\frac{(a+1)^{\left(\frac{\bar{l}-l}{\bar{l}}\right)}}{\ln(a+1)}\bar{l}dl\right\} \\
&= \frac{1}{\bar{l}}\left\{\bar{l}^2 + \frac{\bar{l}^2}{\ln(a+1)} + \left[\frac{(a+1)^{\left(\frac{\bar{l}-l}{\bar{l}}\right)}}{(\ln(a+1))^2}\bar{l}^2\right]_0^{\bar{l}}\right\} \\
&= \frac{1}{\bar{l}}\left\{\bar{l}^2 + \frac{\bar{l}^2}{\ln(a+1)} + \left(\frac{\bar{l}^2}{(\ln(a+1))^2} - \frac{(a+1)}{(\ln(a+1))^2}\bar{l}^2\right)\right\}
\end{aligned}$$

And finally $E_{G^2}(\widetilde{l}) = \bar{l}\left(1+\frac{1}{\ln(a+1)}\left(1-\frac{a}{\ln(a+1)}\right)\right)$, which is positive because $a \leq 1$.

- **The variance $V_{G^2}(\widetilde{l})$**

We have $V_{G^2}(\widetilde{l}) = \int\limits_0^{\bar{l}} \left(l - \bar{l}\left(1 + \frac{1}{\ln(a+1)}\left(1 - \frac{a}{\ln(a+1)}\right)\right)\right)^2 dl$.

Let us adopt the following notation: $j = \frac{\bar{l}}{\ln(a+1)}\left(1 - \frac{a}{\ln(a+1)}\right)$. Thus:

$$\begin{aligned} V_{G^2}(\widetilde{l}) &= \int_0^{\bar{l}} \left((l-\bar{l})^2 - 2(l-\bar{l})j + j^2\right) dl \\ &= \int_0^{\bar{l}} \left(l^2 - 2l\bar{l} + \bar{l}^2 - 2(l-\bar{l})j + j^2\right) dl \\ &= \left[\frac{l^3}{3} - l^2\bar{l} + l\bar{l}^2 - l^2 j + 2l\bar{l}j + lj^2\right]_0^{\bar{l}} \\ &= \frac{\bar{l}^3}{3} + \bar{l}^2 j + \bar{l}j^2 \end{aligned}$$

And finally $V_{G^2}(\widetilde{l}) = \bar{l}^3\left(\frac{1}{3} + m + m^2\right)$, with $m = \frac{1}{\ln(a+1)}\left(1 - \frac{a}{\ln(a+1)}\right)$. ♦

Figure 1

Density f [1] when Revenues Vary

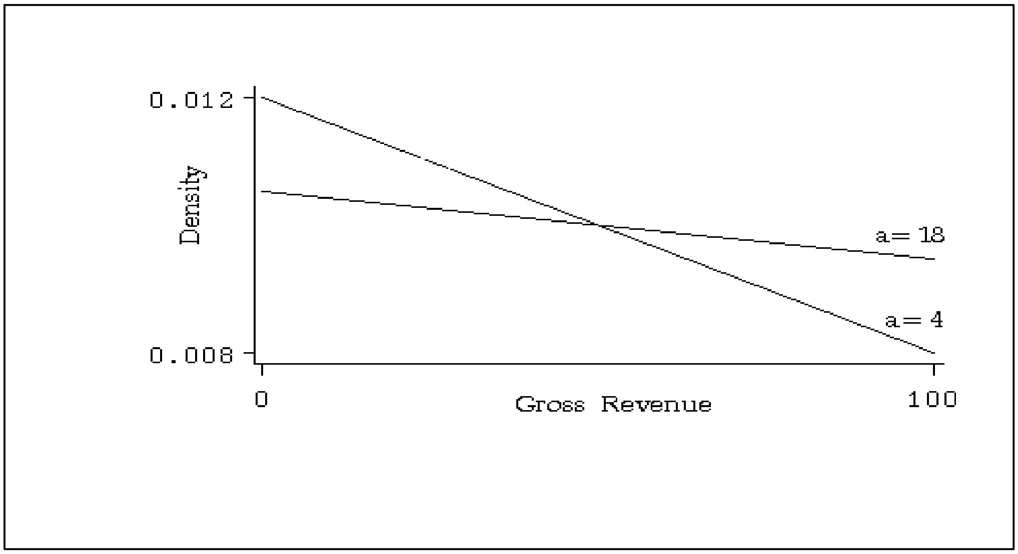


Figure 2

Density f [2] when Losses Vary

For k = 700

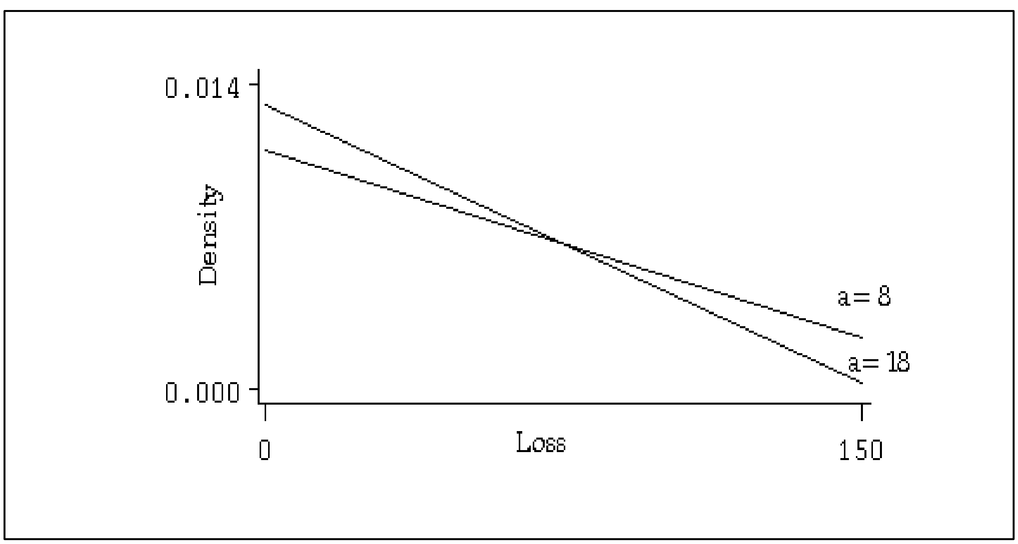


Figure 3

Density g^1 when Revenues Vary

For k = 2

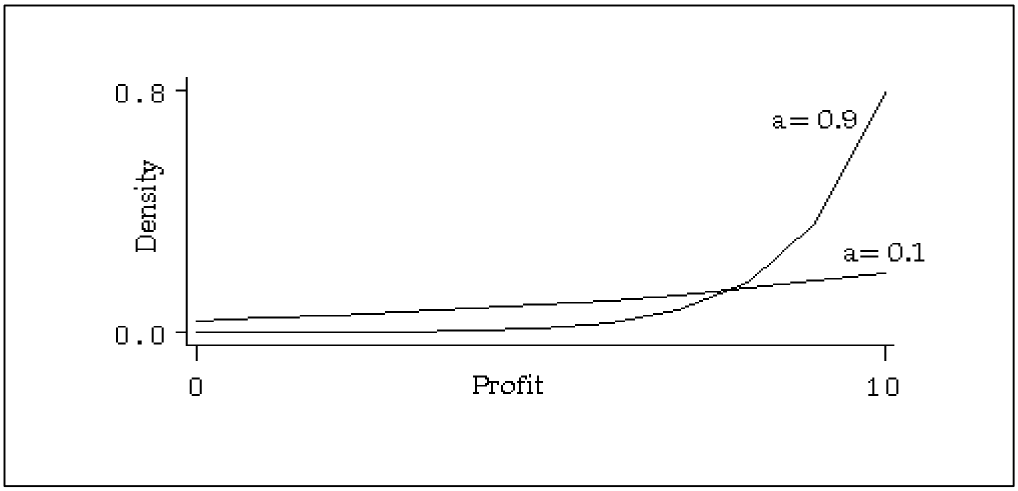
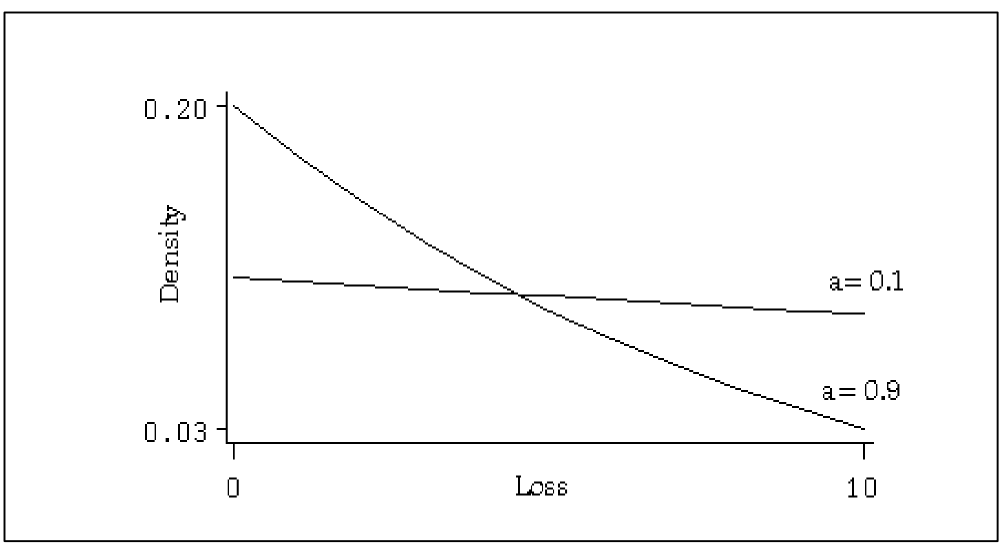


Figure 4

Density g^2 when Losses Vary



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