Proper Risk Behavior

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Abstract

How does risk aversion affect choices when expenses improve probabilities? Attempts to answer this question in the literature found an endogenous switching probability. In this paper we introduce a new concept of comparative attitude to risk, namely proper risk behavior and determine ½ as the threshold probability over which a more proper risk behavior agent becomes a gambler. We consider applications related to self-protection and willingness to pay. We give a sufficient condition for analytic comparative proper risk behavior and show that all results hold in the presence of a background risk.

Keywords: Mixed risk aversion, proper risk aversion, proper risk behavior, self-protection, willingness to pay, background risk, principal-agent.

Résumé

Comment la riscophobie peut-elle affecter les choix lorsque les dépenses affectent les probabilités ? Des essais pour répondre à cette question ont proposé une probabilité endogène cible. Dans cette étude, nous introduisons un nouveau concept, soit le comportement cohérent face au risque, et déterminons ½ comme étant la probabilité cible au-delà de laquelle un agent qui a un comportement plus cohérent devient un joueur. Nous considérons également des applications reliées à la prévention et à la volonté à payer. Nous proposons une condition suffisante pour comparer les divers degrés de comportement cohérents et nous montrons comment nos résultats peuvent être étendus à des situations avec deux sources de risques.

Mots clés : Aversion au risque mélangée, aversion au risque cohérente, comportement cohérent face au risque, autoprotection, volonté à payer, principal agent.

1 Introduction

For many economic applications under risk and uncertainty, a simple concave transformation of a von Newmann-Morgenstern utility function (or an Arrow-Pratt increase in risk aversion) does not yield intuitive changes in decision variables or in lottery choices by risk averse individuals. For example, Ross [1981] showed that the risk premium of a more risk averse agent may not be larger than that of a less risk averse agent in the presence of a background risk or that a more risk averse individual may choose a more risky portfolio in the same environment.

In another example, following the contribution of Ehrlich and Becker [1972] who introduced the concepts of self-protection and self-insurance in the literature, Dionne and Eeckhoudt [1985] showed that a more risk averse individual does not necessarily produce more self-protection activities than a less risk averse one[1]. In fact, one cannot make any prediction on how a more risk averse agent will choose his optimal level of effort in a principal-agent relationship without introducing strong assumptions such as the separability of the utility function (Arnott, 1992).

A third example concerns the willingness to pay literature (Drèze, 1962; Jones-Lee, 1974; and Pratt and Zeckhauser, 1996). One can easily verify that a more risk averse decision maker in the sense of Arrow-Pratt is not necessarily willing to pay more for a lower probability of death or for a lower probability of accident than a less risk averse decision maker (Eeckhoudt, Godfroid and Gollier, 1997). In a fourth example, McGuire, Pratt and Zeckhauser [1991] showed that more risk averse individuals may choose more risky decisions (described as less insurance and more gamble) than less risk averse individuals. They veri ed that these behaviors are function of a critical switching probability.

In the three examples discussed in the two preceding paragraphs, the individuals decisions imply rst order shifts instead of pure second order ones. Moreover, as we will see, their actions usually affect higher moments when appropriate restrictions are not imposed[2]. Consequently, to make predictions

[1] On this issue see also Briys and Schlesinger [1990], Julien, Salanié and Salanié [1998] and Chiu [1997].

[2] For the self-protection example, the i^{th} moment of the gross expected loss is $p(x)\,l^i$, where p is the probability of accident, x is the level of self-protection and l is the amount of loss in case of accident.

on (risk averse) decision makers' behaviors, one needs restrictions either on utility functions or on distribution functions that take into account all distribution moments that are modified by the individuals' choices. In this paper, we shall concentrate on restrictions related to utility functions. For an analysis of restrictions on distribution functions see Julien, Salanié and Salanié [1998], and for restrictions on the loss function see Lee [1998].

In 1987, Pratt and Zeckhauser introduced the concept of Proper Risk Aversion in order to make prediction of lottery choices in presence of an independent, undesirable lottery or of an independent background risk. Their concept is preserved in the class of utility functions that are completely monotone or whose derivatives alternate in sign, with positive odd derivatives and negative even derivatives. These functions come from a mixture of risk averse exponential utility functions. Brocket and Golden [1987] developed a parallel characterization of such functions and Hammond [1974] proposed a first application using a mixture (discrete) of exponential functions.

Recently, Caballé and Pomansky [1996] extended the analysis by characterizing stochastic dominance in presence of such functions. They applied their model to the standard portfolio choice and provided a new set of sufficient conditions to obtain that a mixed risk averse individual will decrease his risky position when the risk increases. One can also show that simple concave transformations of mixed risk aversion functions are sufficient to make comparison of different risk averse individual choices for this simple portfolio problem without a background risk and where the decision variable does not affect the mean of the random variable.

However, up to now, no study has proposed a transformation of the utility function that would permit comparison of individual decisions that affect all the moments of the distribution. The objective of this paper is to propose such a transformation for mixtures of exponential utilities.

In Section 2, we discuss on how the concept of mixed risk aversion is useful to compare the levels of self-protection between a risk averse agent and a risk neutral one. We first obtain that there exists an endogenous probability such that a risk averse individual will produce more self-protection activities than a risk neutral one[3]. A more interesting result is to find an exogenous bound for such probability. In fact, under mixed risk aversion, this threshold probability will be shown to be lower than 1/2. We also obtain that the threshold

[3] Jullien, Salanié and Salanié [1998] derived simultaneously and independently an identical result.

probability is equal to 1/2 when the utility function is quadratic. Two direct extensions of these results will imply that the switching probability de ned in Mcguire, Pratt and Zeckhauser [1991] is greater than 1/2 under mixed risk aversion[4] and that the willingness to pay threshold is also lower than 1/2 when we compare the choice of a risk averse individual to that of a risk neutral one.

However, mixed risk aversion is not sufficient to compare such decision variables between different risk averse individuals. In Section 3 we propose the concept of Proper Risk Behavior. We use the term Proper Risk Behavior since our concern is to compare optimal decision variables that affect all the moments of the random variable distribution. We apply this new concept to the class of mixed risk averse functions.

By de nition, individual v has a more proper risk behavior than individual u if he is more risk averse, more prudent, more temperent ... or if the absolute ratio of the n^{th+1} derivative of v over the n^{th} is higher than the corresponding ratio of individual u for all n greater than one. We provide different characteristics of the proper risk behavior function and we obtain that many utility functions share the notion of proper risk behavior.

Among other results, we will show that the threshold probability where individuals having a more proper risk behavior will produce more self-protection activities or will be willing to pay more for lower probabilities of accidents remains lower than 1/2. This result is important since the great majority of risky situations that include self-protection and public decisions on safety are characterized for events with probability lower than 1/2. We also obtain that the switching probability to become a gambler remains greater than 1/2 in the probability-improving environment of McGuire, Pratt and Zeckhauser. Finally, we extend the concept of Proper Risk Behavior to risky situations with a background risk (Doherty and Schlesinger, 1983).

[4] In their model, activity x increases the winning probability instead of decreasing the probability of loss as in the self-protection and willingness to pay applications.

2 Mixed risk aversion and self-protection

In this section we show that the concept of mixed risk aversion is useful to compare the optimal decision of self-protection between a risk neutral agent and a risk averse one.

The standard model for self-protection (Ehrlich and Becker, 1972) can be summarized as follows. Consider an individual with an increasing von-Neuman-Morgenstern utility function u and a non-random initial wealth w_0. The agent faces a risk of total loss l and can invest in self-protection activities an amount x in order to reduce the probability of loss $(p(x))$, a decreasing and convex function of x. Self-protection activities do not necessarily reduce risk since in general they do not reduce the spread of incomes across states as in the insurance choice. With two states of the world the optimal choice of self-protection is solution of:

$$\max_x \; p(x)\, u(w_0 - l - x) + (1 - p(x))\, u(w_0 - x)$$

from which we derive the following rst order condition:

$$\begin{aligned} 0 &= p'(x)\left[u(w_0 - l - x) - u(w_0 - x)\right] - \\ &\quad \left[p(x)\, u'(w_0 - l - x) + (1 - p(x))\, u'(w_0 - x)\right]. \end{aligned} \tag{1}$$

Note that risk aversion is not sufficient for having the second order condition negative (see Arnott, 1992 for details). In the reminder of this paper we assume that all conditions for having the solution of (1) as a global maximum are met. We now compare this optimal solution to that of a risk neutral agent.

Proposition 1 *Let us de ne x_u and x_n as the optimal levels of self-protection respectively for the risk averse individual and for the risk neutral one. Then there exists a threshold probability $\overline{p}$ such that $x_u > x_n$ if and only if $p(x_n) < \overline{p}$, where $\overline{p}$ is de ned as:*

$$\overline{p} = \frac{\sum_{i=1}^{\infty} (-l)^i \dfrac{u^{(i+1)}(w_0 - x_n)}{(i+1)!}}{\sum_{i=1}^{\infty} (-l)^i \dfrac{u^{(i+1)}(w_0 - x_n)}{i!}}.$$

Proof:

The risk neutral individual faces:

$$\min_x [x + p(x) l].$$

With the associated FOC providing x_n:

$$1 + p'(x) l = 0. \tag{2}$$

By properties of the concave functions, we know that there exists a unique $\overline{p}$ which verify:

$$\frac{u(w_0 - x_n) - u(w_0 - l - x_n)}{l} = \overline{p} u'(w_0 - l - x_n) + (1 - \overline{p}) u'(w_0 - x_n). \tag{3}$$

We evaluate (3) at x_n by calculating

$$Sign \left[\begin{array}{c} p'(x_n)[u(w_0 - l - x_n) - u(w_0 - x_n)] - \\ (p(x_n) u'(w_0 - l - x_n) + (1 - p(x_n)) u'(w_0 - x_n)) \end{array} \right]. \tag{4}$$

Using (2) and (3) into (4) at the optimum x_n we have to evaluate:

$$Sign[(p(x_n) - \overline{p})(u'(w_0 - x_n) - u'(w_0 - l - x_n))].$$

Since the marginal utility is decreasing, if $p(x_n) \leq \overline{p}$, we have

$$(p(x_n) - \overline{p})(u'(w_0 - x_n) - u'(w_0 - l - x_n)) \geq 0$$

which implies that $x_n \leq x_u$.

By taking Taylor expansions around $w_0 - x_n$ in equation (3) , one obtains:

$$\sum_{i=1}^{\infty} (-l)^i \frac{u^{(i+1)}(w_0 - x_n)}{(i+1)!} = \overline{p} \sum_{i=1}^{\infty} (-l)^i \frac{u^{(i+1)}(w_0 - x_n)}{i!},$$

or

$$\overline{p} = \frac{\sum_{i=1}^{\infty} (-l)^i \frac{u^{(i+1)}(w_0 - x_n)}{(i+1)!}}{\sum_{i=1}^{\infty} (-l)^i \frac{u^{(i+1)}(w_0 - x_n)}{i!}}. \tag{5}$$

■

In Proposition 1 the threshold probability ($\overline{p}$) is endogenous since is depends on outcomes and on preferences. However, we can prove the next theorem that uses mixed risk aversion de ned as:

De nition 1 (*Caballé and Pomansky,* 1996) *A real-valued continuous utility function* u *de ned on* $[0, \infty)$ *exhibits mixed risk aversion if and only if it has a completely monotone rst derivative on* $(0, \infty)$ $\left((-1)^{n+1} u^{(n)}(w) \geq 0\right.$, *for* $\left.n \geq 1\right)$, *and* $u(0) = 0$.

We then have:

Theorem 1 *If* u *is mixed risk averse, then* $x_u \geq x_n$ *only if the probability of loss resulting from the optimal choice of the risk neutral agent* $(p(x_n))$ *is lower than* $1/2$.

From Theorem 1 we know that if the optimal choice of self-protection expenses for a risk neutral agent is done and if the probability of loss evaluated at this optimal level is higher than 1/2, then any risk neutral agent will spend more in self-protection activities than does any mixed risk averse agent.

Proof of Theorem 1:

We need to prove that $\overline{p} \leq 1/2$. Since $(i+1)! = (i+1)\, i! > 2i!$ we have

$$\frac{1}{(i+1)!} < \frac{1}{2}\frac{1}{i!},$$

and $(-l)^i u^{(i+1)}(w_0 - x) \geq 0$, then

$$(-l)^i \frac{u^{(i+1)}(w_0 - x)}{(i+1)!} \leq \frac{1}{2}(-l)^i \frac{u^{(i+1)}(w_0 - x)}{i!}.$$

Taking the summation over $i \geq 1$ gives:

$$\sum_{i=1}^{\infty} (-l)^i \frac{u^{(i+1)}(w_0 - x)}{(i+1)!} \leq \frac{1}{2}\sum_{i=1}^{\infty} (-l)^i \frac{u^{(i+1)}(w_0 - x)}{i!},$$

then by (5)

$$\overline{p} \leq 1/2.$$

■

In the case of a quadratic utility function, even if it does not belong to the class of complete monotone functions, we have a more precise value for $\overline{p}$.

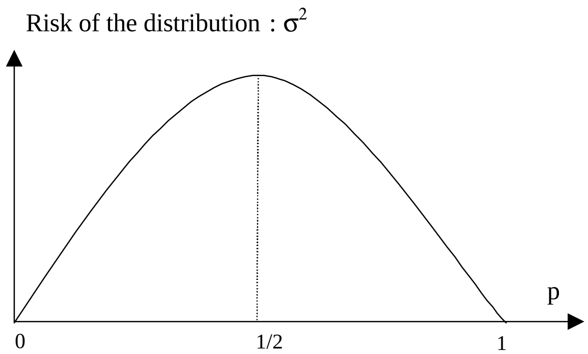

Figure 1:

Proposition 2 *If u exhibits a quadratic utility function, then the threshold probability $\overline{p}$ is equal to* $1/2$.

In this particular case, all derivatives higher than two are nil and $\overline{p}$ as de ned in Proposition 1 is reduced to:

$$\overline{p} = \frac{-l \dfrac{u^{(2)}(w_0 - x_n)}{2!}}{-l \dfrac{u^{(2)}(w_0 - x_n)}{1!}} = 1/2.$$

The graphical representation in Figure 1 illustrates clearly the intuition behind Proposition 2 where the variance is the exact measure of risk.

The risk of the gross expected loss is reduced for the quadratic case to the variance $(\sigma^2 = p(x)(1 - p(x)) l^2)$. As we evaluate the optimal condition of the risk averse individual at the optimal level of self protection of the risk neutral agent there is no mean effect. For all probabilities lower than $1/2$, an increase in the level of self protection decreases p and decreases the variance. However, when the initial probability is larger than $1/2$, an increase in the level of self-protection increases the variance which reduces the welfare of the agent.

3 Proper risk behavior

3.1 Definitions

Caballé and Pomansky [1996] generalized the Arrow-Pratt index of absolute risk aversion to higher order. They defined the n^{th} order index of absolute risk aversion as

$$A_n^u(w) = -\frac{u^{(n+1)}(w)}{u^{(n)}(w)}, \text{ for } n \geq 1.$$

Kimball [1993] introduced the concept of standard risk aversion and showed that the definition is equivalent to decreasing absolute risk aversion and decreasing absolute prudence. We now consider the monotonicity of A_n^u and its implication on decisions variables that affect all moments of distribution functions. Let us introduce the following definition.

Definition 2 *u has a proper risk behavior if and only if $A_n^u(w)$ is decreasing in w for all n.*

Definition 2 is a generalization of mixed risk aversion. In fact as we will see in the next proposition mixed risk aversion utility functions are among the class of proper risk behavior functions, the reverse is not always true. In the reminder of this article we apply the concept of proper behavior to mixed risk averse utility functions. As pointed out by Pratt and Zeckhauser [1987], most known utility functions that are commonly used in economics and finance such as the logarithmic and the power functions are in the class of completely monotone functions. (See Brocket and Golden, 1987 for other examples). These functions have the property of being characterized by the measure describing the mixture of exponential utilities:

For any mixed risk averse utility function $u(w)$, there exists a distribution function F_u satisfying

$$\int_1^\infty \frac{dF_u(t)}{t} < \infty$$

with

$$u(w) = \int_0^\infty \frac{1 - e^{-tw}}{t} dF_u(t).$$

We now have the next result:

Proposition 3 *The next assertions are equivalent and are veri ed for all mixed risk averse utility function u,*

i) $A_n^u(.)$ is decreasing in w for all n.

ii) $A_n^u(w) \leq A_{n+1}^u(w)$ for all w and n.

Proof of Proposition 3:

The equivalence between *i)* and *ii)* is immediate since

$$Sign\left(\frac{d}{dw}A_n^u(w)\right) = Sign\left(A_{n+1}^u(w) - A_n^u(w)\right).$$

If u is mixed risk averse as described by the distribution function F_u, *ii)* is equivalent to

$$\frac{\int_0^\infty t^{n-1}e^{-wt}dF_u(t)}{\int_0^\infty t^{n-2}e^{-wt}dF_u(t)} \leq \frac{\int_0^\infty t^{n}e^{-wt}dF_u(t)}{\int_0^\infty t^{n-1}e^{-wt}dF_u(t)}. \tag{6}$$

To prove (6) we apply Gauchy-Schwartz inequality, i.e.,

$$\left(\int \varphi(t)\psi(t)\,dF(t)\right)^2 \leq \int \varphi^2(t)\,dF(t)\int \psi^2(t)\,dF(t),$$

to

$$\varphi(t) = t^{n/2}e^{-\omega t/2},\ \ \psi(t) = t^{(n-2)/2}e^{-\omega t/2}.$$

■

3.2 Comparative proper risk behavior

Let us consider two risk averse agents u and v. Following Pratt [1964], it has been established that comparative risk aversion can be reduced to applying a simple concave transformation of utility functions: v is more risk averse than u if and only if $v = k(u)$ with $k'' < 0$. Such transformation is not sufficient to obtain a more proper risk behavior function. Let us consider the next de nition:

De nition 3 *Let u and v be two proper risk behavior utility functions. We say that v has a more proper risk behavior than u if and only if $A_n^u(w) \leq A_n^v(w)$, for all n and w.*

We will show how this new de nition is useful to make comparison of decision variables that affect all distribution moments between individuals having different proper risk behavior.

We have the next transformation theorem.

Theorem 2 *Let u and v be two mixed risk averse utility functions described respectively by distribution functions F_u and F_v. If $\dfrac{dF_u(.)}{dF_v(.)}$ is decreasing over $(0,\infty)$, then v has a more proper risk behavior than u.*

Before presenting the proof, let us consider a concrete example:

$$\begin{aligned} u(w) &= -p_1 e^{-a_1 w} - p_2 e^{-a_2 w} - ... - p_n e^{-a_n w}, \\ v(w) &= -q_1 e^{-a_1 w} - q_2 e^{-a_2 w} - ... - q_n e^{-a_n w} \end{aligned}$$

with p_i, q_i, and a_i as positive parameters for $i = 1, ...n$ and $a_1 < a_2 < ... < a_n$. If $\frac{p_1}{q_1} \geq ... \geq \frac{p_n}{q_n}$, then from Theorem 2 we know that v is more risk averse, more prudent, more temperent than u, and more generally

$$A_n^u(.) \leq A_n^v(.), \text{ for all } n \geq 1.$$

The intuition behind Theorem 2 is quite simple. Consider for the sake of illustration the case where there are only two positive a_i (a_1 and a_2) in the example above. By transforming p_1 into q_1 lower than p_1, less weight is put upon the less risk averse component of the u function (since $a_1 < a_2$). Of course lowering p_1 also implies that q_2 exceeds p_2 so that simultaneously more weight is placed upon the more risk averse component of u. As result v is surely more risk averse than u and the theorem shows that this property automatically extends to all ratios of successive derivatives of each utility function.

Proof of Theorem 2:

We need to show that

$$\frac{\int_0^\infty t^{n+1} e^{-wt} dF_u(t)}{\int_0^\infty t^n e^{-wt} dF_u(t)} \leq \frac{\int_0^\infty t^{n+1} e^{-wt} dF_v(t)}{\int_0^\infty t^n e^{-wt} dF_v(t)}, \text{ for } n \geq 0.$$

The latter is equivalent to

$$\int_0^\infty t d\widetilde{F}_u^{n,w}(t) \leq \int_0^\infty t d\widetilde{F}_v^{n,w}(t), \tag{7}$$

where

$$d\widetilde{F}_i^{n,w}(t) = \frac{t^n e^{-wt} dF_i(t)}{\int_0^\infty s^n e^{-ws} dF_i(s)}, \text{ for } i = u,\ v.$$

Inequality (7) is equivalent to[5]

$$\int_0^\infty [1 - \widetilde{F}_u^{n,w}(t)]dt \leq \int_0^\infty [1 - \widetilde{F}_v^{n,w}(t)]dt,$$

or

$$\int_0^\infty [\widetilde{F}_v^{n,w}(t) - \widetilde{F}_u^{n,w}(t)]dt \leq 0. \tag{8}$$

A sufficient condition to have (8) is to show that

$$\widetilde{F}_v^{n,w}(t) \leq \widetilde{F}_u^{n,w}(t), \text{ for all } t \text{ and all } n. \tag{9}$$

For a complete monotone function the last inequality simpli es to

$$\frac{\int_0^t s^n e^{-ws} dF_u(s)}{\int_0^t s^n e^{-ws} dF_v(s)} \geq \frac{\int_0^\infty s^n e^{-ws} dF_u(s)}{\int_0^\infty s^n e^{-ws} dF_v(s)}.$$

Since

$$\frac{\int_0^\infty s^n e^{-ws} dF_u(s)}{\int_0^\infty s^n e^{-ws} dF_v(s)} = \lim_{t\to\infty} \frac{\int_0^t s^n e^{-ws} dF_u(s)}{\int_0^t s^n e^{-ws} dF_v(s)},$$

the result will be done if we prove that

$$K(t) = \frac{\int_0^t s^n e^{-ws} dF_u(s)}{\int_0^x s^n e^{-ws} dF_v(s)}$$

is decreasing in t.

A simple calculation shows that $K(.)$ is decreasing if and only if

$$\frac{dF_u(t)}{dF_v(t)} \leq K(t). \tag{10}$$

We now prove (10):

$$\begin{aligned} K(t) &= \frac{\int_0^t s^n e^{-ws} \frac{dF_u(s)}{dF_v(s)} dF_v(s)}{\int_0^t s^n e^{-ws} dF_v(s)} \\ &\geq \frac{dF_u(t)}{dF_v(t)} \frac{\int_0^t s^n e^{-ws} dF_v(s)}{\int_0^t s^n e^{-ws} dF_v(s)} = \frac{dF_u(t)}{dF_v(t)}. \end{aligned}$$

This completes the proof of Theorem 2. ■

[5] See Lemma 1, page 148 in Feller, 1971.

4 Applications

4.1 Self-Protection

As shown by Briys and Schlesinger [1990], self-protection activities do not necessarily reduce risk, but affects the probabilities of the various states as well as the contingent outcomes. The problem here is different from that where probabilities are xed as in the context of market insurance. One consequence is that more risk averse agents do not necessarily choose higher level of self-protection spending (see Dionne and Eeckhoudt, 1985 for explicit examples). McGuire, Pratt and Zeckhauser [1991] found an endogenous critical switching probability that depends on preferences and outcomes, and interpret expenditures as gambling or insurance. This endogenous switching probability is retrieved by Lee [1998]. In this section, we show that the endogenous probability is lower than 1/2 for self-protection and willingness to pay and greater than 1/2 in the probability improving environment of McGuire, Pratt and Zeckhauser [1991].

Proposition 4 *Assume v is more risk averse than u in the sense of Arrow-Pratt, then there exists a threshold probability $\overline{p}$ such that self-protection is higher for v than for u if and only if the probability of loss resulting from the optimal choice of u is less than $\overline{p}$, with*

$$\overline{p} = \frac{\sum_{i=1}^{\infty} \frac{(-1)^i}{i!} l^i \left[v' u^{(i)} - u' v^{(i)} \right]}{\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} (-1)^i (-1)^j \frac{l^i l^j}{i! j!} \left[v^{(i)} u^{(j+1)} - u^{(i)} v^{(j+1)} \right]}.$$

Proof:

We evaluate the FOC for agent v at x_u. After simpli cation we obtain $x_u \leq x_v$ if and only if

$$\frac{p(x_u)}{1 - p(x_u)} \leq \frac{v'(w_0 - x_u)\Delta u - u'(w_0 - x_u)\Delta v}{u'(w_0 - x_u - l)\Delta v - v'(w_0 - x_u - l)\Delta u},$$

where

$$\Delta u = u(w_0 - x_u - l) - u(w_0 - x_u), \ \Delta v = v(w_0 - x_u - l) - v(w_0 - x_u).$$

Since $\dfrac{p}{1-p}$ is strictly increasing and maps $(0,1)$ into $(0,\infty)$, we know that there exists a unique $\overline{p}$ verifying

$$\frac{\overline{p}}{1-\overline{p}} = \frac{v'\left(w_0 - x_u\right)\Delta u - u'\left(w_0 - x_u\right)\Delta v}{u'\left(w_0 - x_u - l\right)\Delta v - v'\left(w_0 - x_u - l\right)\Delta u},$$

which simpli es to

$$\overline{p} = \frac{v'\left(w_0 - x_u\right)\Delta u - u'\left(w_0 - x_u\right)\Delta v}{\Delta u' \Delta v - \Delta v' \Delta u}. \tag{11}$$

Taylor expansion around $w - x$ gives

$$\Delta u = \sum_{i=1}^{\infty} (-1)^i \frac{l^i}{i!} u^{(i)}\left(w_0 - x_u\right), \;\; \Delta v = \sum_{i=1}^{\infty} (-1)^i \frac{l^i}{i!} v^{(i)}\left(w_0 - x_u\right)$$

$$\Delta u' = \sum_{i=1}^{\infty} (-1)^i \frac{l^i}{i!} u^{(i+1)}\left(w_0 - x_u\right), \;\; \Delta v' = \sum_{i=1}^{\infty} (-1)^i \frac{l^i}{i!} v^{(i+1)}\left(w_0 - x_u\right).$$

We can rewrite (11) as:

$$\overline{p} = \frac{\displaystyle\sum_{i=1}^{\infty} \frac{(-1)^i}{i!} l^i \left[v' u^{(i)} - u' v^{(i)}\right]}{\displaystyle\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} (-1)^i (-1)^j \frac{l^i l^j}{i! j!} \left[v^{(i)} u^{(j+1)} - u^{(i)} v^{(j+1)}\right]}. \tag{12}$$

■

As for Proposition 1, $\overline{p}$ given by (12) is still endogenous since it depends on u, v and on outcomes. In Section 2, the concept of mixed risk aversion was applied to compare self-protection activities of a risk neutral individual vs a risk averse one. We now extend the result of Theorem 1 and study the relationship between risk attitude and self-protection spending to the class of proper risk behavior agents. In fact, we can show the next result.

Theorem 3 *let u and v be two mixed risk averse utility functions and x_u, x_v be their corresponding optimal levels of self-protection. If v has a more proper risk behavior than u, then $x_v \geq x_u$ only if the probability of loss resulting from the optimal choice of u is less than $1/2$.*

Theorem 3 says that if the probability of loss resulting from the optimal choice of agent u is higher than 1/2, then whatever is the level of outcomes, agent v will spend less in self-protection activities if he has a more proper risk behavior than u.

Proof of Theorem 3:

It is sufficient to prove that $\overline{p} \leq 1/2$. Let s denote

$$K_{ij} = (-1)^i (-1)^j \left[v^{(i)} u^{(j+1)} - u^{(i)} v^{(j+1)} \right].$$

By Proposition 3 we can show that:

$$K_{ij} \begin{cases} > 0 \text{ if } i < j+1 \\ = 0 \text{ if } i = j+1 \\ < 0 \text{ if } i > j+1 \end{cases}. \tag{13}$$

The denominator in (12) can be written as:

$$-l \sum_{i=1}^{\infty} \frac{(-1)^i}{i!} l^i \left[v' u^{(i+1)} - u' v^{(i+1)} \right] + \sum_{i=2}^{\infty} \sum_{j=1}^{\infty} \frac{l^{i+j}}{i!j!} K_{ij}.$$

Now we prove that

$$\sum_{i=2}^{\infty} \sum_{j=1}^{\infty} \frac{l^{i+j}}{i!j!} K_{ij} \geq 0.$$

In fact, for $i \geq j+1$

$$\begin{aligned} K_{ij} &= (-1)^i (-1)^j \left[v^{(i)} u^{(j+1)} - u^{(i)} v^{(j+1)} \right] \\ &= -(-1)^i (-1)^j \left[u^{(i)} v^{(j+1)} - v^{(i)} u^{(j+1)} \right] \\ &= -(-1)^{i-1} (-1)^{j+1} \left[u^{(i)} v^{(j+1)} - v^{(i)} u^{(j+1)} \right] \\ &= -K_{j+1,i-1}, \end{aligned}$$

since $j+1 \leq (i-1)+1$ then by (13)

$$K_{j+1,i-1} \geq 0.$$

Moreover since $i > j+1$ then

$$\frac{1}{i!j!} < \frac{1}{(j+1)!\,(i-1)!}$$

and consequently

$$\frac{1}{i!j!}K_{ij} + \frac{1}{(j+1)!\,(i-1)!}K_{j+1,i-1} \geq 0.$$

As a result

$$\forall\ n \geq 2,\ \ \sum_{i=2}^{n}\sum_{j=1}^{n}\frac{L^{i+j}}{i!j!}K_{ij} \geq 0,$$

and at the limit we obtain

$$\sum_{i=2}^{\infty}\sum_{j=1}^{\infty}\frac{l^{i+j}}{i!j!}K_{ij} \geq 0.$$

The denominator in (12) is the sum of two positive terms. We can then write

$$\overline{p} \leq \frac{\sum_{i=1}^{\infty}\frac{(-1)^i}{i!}l^i\left[v'u^{(i)} - u'v^{(i)}\right]}{\sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{i!}l^{i+1}\left[v'u^{(i+1)} - u'v^{(i+1)}\right]}$$

or

$$\overline{p} \leq \frac{\sum_{i=2}^{\infty}\frac{(-1)^i}{i!}l^i\left[v'u^{(i)} - u'v^{(i)}\right]}{\sum_{i=2}^{\infty}\frac{(-1)^i}{(i-1)!}l^i\left[v'u^{(i)} - u'v^{(i)}\right]}.$$

Since for $i \geq 2$, $i! > 2\,(i-1)!$, and since $(-1)^i\,l^i\left[v'u^{(i)} - u'v^{(i)}\right] > 0$, we then have

$$\frac{(-1)^i}{i!}l^i\left[v'u^{(i)} - u'v^{(i)}\right] < \frac{1}{2}\frac{(-1)^i}{(i-1)!}l^i\left[v'u^{(i)} - u'v^{(i)}\right],$$

taking the summation over $i \geq 2$ gives $\overline{p} \leq 1/2$. ■

By symmetry it can be obtained that $\overline{p} \geq 1/2$ when $p\,(x)$ is increasing in x as in McGuire, Pratt and Zeckhauser [1991]. The mathematical development is identical to that made in Theorem 3 with $1 - p(x)$ be the new probability of loss for the modi ed problem. We have the next result:

Corollary 1 *When $p\,(x)$ is increasing in x, if v and u are mixed risk averse and if v has a more risk behavior than u, then $x_v \geq x_u$ only if the winning probability resulting from the optimal choice of u is higher than $1/2$.*

4.2 Willingness To Pay

We rst present the formal analysis of Pratt and Zeckhauser [1996]. An individual adhering to the axioms of von Neumann-Morgenstern utility theory has the utility function $u(s, w_0)$, where s is equal to 0 if the individual dies and 1 if the individual survives, and w_0 is initial wealth. The individual faces a probability of death p. His expected utility is then given by

$$U = pu(0, w_0) + (1 - p)u(1, w_0).$$

The agent is given the opportunity to reduce the probability of death from p to $p - r$. He will accept to forfeit a positive amount of wealth in order to reduce this probability. Willingness To Pay (WTP) is then de ned as the maximum amount x that he would pay for such reduction, i.e., x is solution of

$$(p - r)u(0, w_0 - x) + (1 - p + r)u(1, w_0 - x) = U,$$

and for all $x' > x$,

$$(p - r)u(0, w_0 - x') + (1 - p + r)u(1, w_0 - x') < U.$$

Willingness to pay is a guideline for public and private investment policies and according to WTP, public investment projects, such as health care, environment or road safety conditions will be recommended only if the total of sums that different agents bene ciating from favorable probability changes exceeds the capital cost of the project in concern. Alternative resource allocations are also compared on the basis of WTP.

In other situations it is more appropriate to offer different bundles of risk to different individuals if valuation of risk are different among agents. It is then necessary for establishing those bundles to know WTP for the different risk classes. It is this last point that we will discuss in the reminder of this section. Suppose that risk-reducing bene ts are privately valued (medicine expenses). As pointed out by Pratt and Zeckhauser [1996], individuals with high valuation of risk reduction would choose expensive plans which may lead to services that are ineffective to most individuals. As we did for the self-protection model we will use risk aversion to order WTP values and show that higher valuation of risk reduction does not necessarily coincide with higher risk aversion.

Back to the model, we suppose that $u(0, w_0) = u(1, w_0 - l) = u(w_0 - l)$. Let s denote x_u as the WTP of agent u given the probability of loss is reduced from p to $p - r$. We can show the next result.

Theorem 4 *let u and v be two mixed risk averse utility functions and x_u, x_v their corresponding optimal amounts of willingness to pay. If v has a more proper risk behavior than u, then $x_v \geq x_u$ only if the probability of loss resulting from the optimal choice of u is lower than $1/2$.*

Proof:

The expected utility for u is:

$$U = pu\left(w_0 - l\right) + \left(1 - p\right)u\left(w_0\right)$$

and for individual v

$$V = pv\left(w_0 - l\right) + \left(1 - p\right)v\left(w_0\right).$$

In order to obtain the willingness to pay for u we completely differentiate U with respect to p and w_0 to have:

$$WTP_u = \frac{dw_0}{dp} = \frac{u\left(w_0\right) - u\left(w_0 - l\right)}{pu'\left(w_0 - l\right) + \left(1 - p\right)u'\left(w_0\right)}. \tag{14}$$

The same result holds for individual v

$$WTP_v = \frac{dw_0}{dp} = \frac{v\left(w_0\right) - v\left(w_0 - l\right)}{pv'\left(w_0 - l\right) + \left(1 - p\right)v'\left(w_0\right)}. \tag{15}$$

The threshold probability $\overline{p}$ is solution of (14)=(15):

$$\overline{p} = \frac{v'\left(w_0\right)\Delta u - u'\left(w_0\right)\Delta v}{\Delta u'\Delta v - \Delta v'\Delta u}. \tag{16}$$

With Taylor expansion we have

$$\Delta u = \sum_{i=1}^{\infty}\left(-1\right)^i \frac{l^i}{i!}u^{(i)}\left(w_0 - x\right),\ \Delta v = \sum_{i=1}^{\infty}\left(-1\right)^i \frac{l^i}{i!}v^{(i)}\left(w_0 - x\right).$$

We can then rewrite (16) as:

$$\overline{p} = \frac{\sum_{i=1}^{\infty}\frac{\left(-1\right)^i}{i!}l^i\left[v'u^{(i)} - u'v^{(i)}\right]}{\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(-1\right)^i\left(-1\right)^j\frac{l^i l^j}{i!j!}\left[v^{(i)}u^{(j+1)} - u^{(i)}v^{(j+1)}\right]}. \tag{17}$$

The remainder of the proof is the same as that of Theorem 3. ∎

5 Background risk

In this section we consider the case where the individual faces a background risk $(\widetilde{\varepsilon})$ on wealth that is independent from the occurrence of an accident. Let s denote $\widetilde{u}(w) = E_{\varepsilon}(u(w+\varepsilon)) = \int u(w+\varepsilon)\, dF(\varepsilon)$. We know that an individual with utility function u and a background risk $\widetilde{\varepsilon}$ behaves as an individual with utility function $\widetilde{u}$ and no background risk. Kimball [1993] showed that if u has a decreasing absolute risk aversion and a decreasing absolute temperance, then these properties hold for $\widetilde{u}$. In other words if u is standard risk averse then $\widetilde{u}$ is also standard risk averse. If we suppose that u is mixed risk averse, then $\widetilde{u}$ is also mixed risk averse and then by Proposition 3 $\widetilde{u}$ has a proper risk behavior i.e., for all $n \geq 1$, $-\frac{\int u^{(n+1)}(w+\varepsilon)dF(\varepsilon)}{\int u^{(n)}(w+\varepsilon)dF(\varepsilon)}$ is decreasing in w, which is a generalization of Proposition 4 in Kimball [1993].

Pratt [1988] showed that provided either u or v has a decreasing absolute risk aversion, $\widetilde{v}$ is more risk averse that $\widetilde{u}$ whenever v is more risk averse than u. The next theorem extends this results to the concept of proper risk behavior for mixed risk averse utility functions.

Theorem 5 *Let u and v be two mixed risk averse functions and suppose that v has a more proper risk behavior than u, then $\widetilde{u}$ and $\widetilde{v}$ are mixed risk averse functions and $\widetilde{v}$ has a more proper risk behavior than $\widetilde{u}$.*

Proof of Theorem 5:

To simplify the notation and without affecting the proof we write ε for $w+\varepsilon$. We need to prove that for all n and ε and whenever $A_n^u(w) \leq A_n^v(w)$, the following inequality holds:

$$-\frac{\int u^{(n+1)}(\varepsilon)\, dF(\varepsilon)}{\int u^{(n)}(\varepsilon)\, dF(\varepsilon)} \leq -\frac{\int v^{(n+1)}(\varepsilon)\, dF(\varepsilon)}{\int v^{(n)}(\varepsilon)\, dF(\varepsilon)},$$

or

$$\int\int v^{(n)}(\varepsilon')\,[u^{(n+1)}(\varepsilon) - u^{(n)}(\varepsilon)\frac{v^{(n+1)}(\varepsilon')}{v^{(n)}(\varepsilon')}]dF(\varepsilon)\, dF(\varepsilon') \geq 0. \qquad (18)$$

Since v has more proper behavior than u, we have

$$-\frac{v^{(n+1)}(\varepsilon')}{v^{(n)}(\varepsilon')} \geq -\frac{u^{(n+1)}(\varepsilon')}{u^{(n)}(\varepsilon')}, \text{ for all } n \text{ and } \varepsilon'$$

and then (since $v^{(n)}(\varepsilon')\,u^{(n)}(\varepsilon) \geq 0$)

$$\begin{aligned}
& v^{(n)}(\varepsilon')\,[u^{(n+1)}(\varepsilon) - u^{(n)}(\varepsilon)\,\frac{v^{(n+1)}(\varepsilon')}{v^{(n)}(\varepsilon')}] \\
\geq \; & v^{(n)}(\varepsilon')\,[u^{(n+1)}(\varepsilon) - u^{(n)}(\varepsilon)\,\frac{u^{(n+1)}(\varepsilon')}{u^{(n)}(\varepsilon')}] \\
= \; & \frac{v^{(n)}(\varepsilon')}{u^{(n)}(\varepsilon')}[u^{(n+1)}(\varepsilon)\,u^{(n)}(\varepsilon') - u^{(n)}(\varepsilon)\,u^{(n+1)}(\varepsilon')].
\end{aligned}$$

The integral in (18) is then superior to

$$\begin{aligned}
& \int\int_{\{\varepsilon\leq\varepsilon'\}} \frac{v^{(n)}(\varepsilon')}{u^{(n)}(\varepsilon')}[u^{(n+1)}(\varepsilon)\,u^{(n)}(\varepsilon') - u^{(n)}(\varepsilon)\,u^{(n+1)}(\varepsilon')]dF(\varepsilon)\,dF(\varepsilon') \\
& + \int\int_{\{\varepsilon\geq\varepsilon'\}} \frac{v^{(n)}(\varepsilon')}{u^{(n)}(\varepsilon')}[u^{(n+1)}(\varepsilon)\,u^{(n)}(\varepsilon') - u^{(n)}(\varepsilon)\,u^{(n+1)}(\varepsilon')]dF(\varepsilon)\,dF(\varepsilon') \\
= \; & -\int\int_{\{\varepsilon\leq\varepsilon'\}} \frac{v^{(n)}(\varepsilon')}{u^{(n)}(\varepsilon')}[u^{(n)}(\varepsilon)\,u^{(n+1)}(\varepsilon') - u^{(n+1)}(\varepsilon)\,u^{(n)}(\varepsilon')]dF(\varepsilon)\,dF(\varepsilon') \\
& + \int\int_{\{\varepsilon\geq\varepsilon'\}} \frac{v^{(n)}(\varepsilon')}{u^{(n)}(\varepsilon')}[u^{(n+1)}(\varepsilon)\,u^{(n)}(\varepsilon') - u^{(n)}(\varepsilon)\,u^{(n+1)}(\varepsilon')]dF(\varepsilon)\,dF(\varepsilon') \\
= \; & -\int\int_{\{\varepsilon\geq\varepsilon'\}} \frac{v^{(n)}(\varepsilon)}{u^{(n)}(\varepsilon)}[u^{(n+1)}(\varepsilon)\,u^{(n)}(\varepsilon') - u^{(n)}(\varepsilon)\,u^{(n+1)}(\varepsilon')]dF(\varepsilon)\,dF(\varepsilon') \\
& + \int\int_{\{\varepsilon\geq\varepsilon'\}} \frac{v^{(n)}(\varepsilon')}{u^{(n)}(\varepsilon')}[u^{(n+1)}(\varepsilon)\,u^{(n)}(\varepsilon') - u^{(n)}(\varepsilon)\,u^{(n+1)}(\varepsilon')]dF(\varepsilon)\,dF(\varepsilon') \\
= \; & \int\int_{\{\varepsilon\geq\varepsilon'\}} \left(\frac{v^{(n)}(\varepsilon')}{u^{(n)}(\varepsilon')} - \frac{v^{(n)}(\varepsilon)}{u^{(n)}(\varepsilon)}\right) \\
\times \; & [u^{(n+1)}(\varepsilon)\,u^{(n)}(\varepsilon') - u^{(n)}(\varepsilon)\,u^{(n+1)}(\varepsilon')]dF(\varepsilon)\,dF(\varepsilon'). \qquad (19)
\end{aligned}$$

Again since v has a more proper risk behavior than u, the ratio $\frac{v^{(n)}(\varepsilon')}{u^{(n)}(\varepsilon')}$ is decreasing; and since -$\frac{u^{(n+1)}}{u^{(n)}}$ is decreasing then

$$u^{(n+1)}(\varepsilon)\,u^{(n)}(\varepsilon') - u^{(n)}(\varepsilon)\,u^{(n+1)}(\varepsilon') \geq 0, \text{ for } \varepsilon' \leq \varepsilon.$$

The last integral in (19) is then positive which ends the proof. ■

Using the result in Theorem 5 we can directly extend the results of Section 4 to situations with a background risk. For example if the probability of loss resulting from the optimal self-protection choice of agent u is lower than $1/2$, and if an agent v has a more proper risk behavior than u, then even in the presence of a background risk, it follows from direct extensions of Theorems 3 and 5 that $\widetilde{x}_v \geq \widetilde{x}_u$, where $\widetilde{x}$ is the optimal level of self-protection in the presence of a background risk.

6 Conclusion

In this article we have proposed the concept of proper risk behavior. We have shown how this extension of proper risk aversion can be useful to make comparison of decision variables that affect all distribution moments between different individuals. We have obtained that mixed risk aversion utility functions are among the class of proper risk behavior functions and consequently proper risk behavior can be applied to mixtures of exponential utility functions that are often discussed in the nance and insurance literatures.

Many extensions of this article can be considered. First it would be interesting to analyze how proper risk behavior functions can be useful to make prediction of the agent s action in a principal-agent framework when utility functions are not additively separable. How different proper risk behavior agents choose the optimal level of effort in function of a given risk sharing contract? A more difficult question would be to compare different risk sharing contracts between different proper risk behavior agents.

Another extension is related to the willingness to pay literature. Up to now, since it was not possible to know the circumstance where it was possible to make comparison of different willingness to pay amounts between different risk averse individuals, the aggregation of such amounts was not possible to implement. Such aggregation is now possible for the class of proper risk behavior utility functions in situations where accidents probabilities are lower than $1/2$. In fact, the class of accidents corresponds to the great majority of observed accidents in the real world.

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