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Essays on Credit Risk and Portfolio Choice

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Résumé

Durant la dernière décennie, la nouvelle réglementation du secteur bancaire a mené à un recours massifs aux notations financières. À titre d'exemple, on peut citer l'ensemble de recommandations stipulées par le Comité de Bâle sur le Contrôle Bancaire. Également, le modèle CreditMetrics, développé par la banque d'affaire JP Morgan en 1997, fait usage de ces notations financières pour la mesure du risque de crédit. Ainsi, on a pu assister à une amplification de l'activité de notation : un plus grand nombre de dettes et d'entreprises est noté par les agences spécialisées à l'instar de Moody's et Standard & Poor's. Une telle tendance sera bénéfique pour les prochaines études empiriques portant sur le sujet.

Généralement, on analyse l'évolution du risque de crédit d'un ensemble des dettes via les matrices de transition. Ces matrices indiquent, par classe de risque, les probabilités de passage d'une note à une autre durant un horizon de temps déterminé. Ainsi, une estimation efficace de ces probabilités est cruciale pour la mesure de la Valeur à Risque d'un portefeuille d'obligations ou l'évaluation des dérivés de crédit.

Au premier essai, on développe une nouvelle technique d'estimation des matrices de transition basée sur le théorème de Bayes. Cette nouvelle technique nous permet de conclure que la probabilité de défaut est strictement positive pour toutes les classes de risque et pour tout horizon de temps considéré. Également, il sera aisé de définir les intervalles de confiance de toute probabilité de transition grâce à cette estimation Bayesienne : une caractéristique permettant de répondre aux impératifs de la réglementation qui exige des tests hors échantillon.

Les probabilités de défaut retrouvées par l'estimation Bayesienne sont utilisées pour la détermination de l'écart de crédit (spread) expliqué par le défaut, en appliquant le modèle développé par Dionne, Gauthier, Hammami, Maurice et Simonato (2005). Nos résultats démontrent que ces écarts de crédit sont supérieurs à ceux obtenus via l'estimation cohorte de la matrice de transition, pour des dettes à courte maturité.

Le deuxième essai porte sur la corrélation des migrations. Une mauvaise estimation de cette corrélation peut induire une sous-évaluation du capital économique d'une banque et ainsi nuire à sa stabilité financière. Également, Duffie, Kapadia et Saita (2007) ont rappelé l'intérêt de bien analyser la corrélation des défauts afin d'évaluer certains actifs financiers tels que les swaps de défaut sur panier (basket default swaps). Ainsi, une estimation efficiente de la corrélation des migrations est nécessaire pour la mesure du risque de crédit.

Cependant, l'estimation et le report des 3136 corrélations de migration possibles en considérant 8 classes de risque ne seraient pas pratiques pour un gestionnaire de portefeuille. Ainsi, on adopte la méthodologie de Jafry et Schuermann (2004) qui consiste à calculer un indice de mobilité résumant l'ensemble des probabilités de transition d'une matrice. À partir des séries temporelles de ces indices de mobilité pour chacun des secteurs économiques aux U.S.A, on vérifie l'existence d'un phénomène de transmission des crises au sein du même secteur. La transmission des crises à travers les secteurs est étudiée par la suite en se basant sur un modèle VAR avec changement de régime. Nos résultats démontrent l'existence de deux régimes (faible et forte corrélation) ainsi qu'un phénomène de contagion entre certains secteurs. À titre d'exemple, la dégradation des notes financières au secteur industriel américain durant le régime de forte corrélation induit une dégradation des notes du secteur bancaire durant le prochain trimestre.

Au troisième essai, on analyse l'énigme du choix de portefeuille proposée par Canner, Mankiw et Weil (1997). L'idée est de tester une conclusion de Elton et Gruber (2000) stipulant qu'un ratio obligations/actions décroissant en fonction de la tolérance face au risque n'implique pas nécessairement une contradiction par rapport à la théorie moderne de choix de portefeuille et n'introduit pas de doute sur la rationalité des choix individuels. À partir de données de 470 portefeuilles individuels d'une entreprise de courtage, on vérifie la pertinence de la mesure de tolérance au risque utilisée par Canner et al.(1997). Ensuite, on obtient que le ratio obligations/actions décroît en relation avec la tolérance face au risque. Finalement, on vérifie l'existence du théorème de séparation à deux fonds dans les données sur les actifs disponibles aux investisseurs de notre échantillon.

Mots clés : Notation financière, matrices de transition, estimation Bayesienne, spread de défaut, indice de mobilité, contagion, rationalité de l'investisseur, énigme du choix de portefeuille, tolérance au risque, théorème de séparation, ratio obligations/actions.

Abstract

Bank regulation urged an increasing use of credit ratings during last decade. For instance, we may cite the package of rules to assess the required capital as recommended by the Basel Committee on Banking Supervision. Moreover, the CreditMetrics model, developed in 1997 by JP Morgan, utilizes these credit ratings as a key input to measure credit risk. As a result, we notice an increase of the credit rating activity: credit rating agencies such as Moody's or Standard & Poor's, assign ratings for a higher number of issues. Thus, more valuable and accurate information could be obtained from these larger data samples.

Usually, the credit risk dynamics are captured via transition matrices. A credit rating transition matrix corresponds to a summary of probabilities for a particular rating to migrate to other ratings within a period of time. Thus, an accurate estimation of these probabilities is needed to measure the Credit Value at Risk of bonds portfolio or to price defaultable securities and credit derivatives.

In the first essay, we develop a new method for estimating the rating transition matrix based on Bayes theorem. We show that default probabilities are non-zero even for the highest rated classes and short maturities. Besides, the Bayesian technique allows us to derive confidence intervals for the transition probabilities. Such ability conforms the regulatory concerns about out of sample testing. Then, we use our Bayesian default probabilities to determine the corporate bond spreads explained by default risk. We adopt the same methodology as described in Dionne, Gauthier, Hammami, Maurice, and Simonato (2005) to compute the default spreads. Our results show that the default spreads are higher than those obtained by cohort technique for short maturities.

The second essay deals with migration correlation. An omission or an inaccurate estimation of that correlation will induce a misestimation of the regulatory capital and, consequently, will embrittle the company's financial stability. Moreover, Das, Duffie, Kapadia, and Saita (2007) document that default correlation analysis is

needed for asset pricing such as basket default swaps. Thus, an efficient estimation of the migration correlation is necessary to assess the credit risk.

However, reporting all migration correlations seems confusing: 3136 migration correlations should be estimated if we consider 8 credit rating classes. Thus, we follow Jafry and Schuermann (2004) to summarize the rating transition matrix into a scalar (a mobility index). Once time series of theses indices are obtained for each U.S. business sector, we check if the crisis transmission phenomenon exists within each business sector. Then, we test for possibly crisis transmission phenomenon among sectors by estimating a Markov Switching Vector Autoregressions model. The results obtained provide evidence of high and low correlation regimes and prove default contagion among some sectors. For example, more downgrades in the U.S. industrial sector during the high correlation regime imply more downgrades in the U.S. banking sector during the next three months.

The third essay proposes an empirical test to the asset-allocation puzzle posed by Canner, Mankiw, and Weil (1997). These authors conclude that the recommendations of some financial advisors are inconsistent with optimal allocation as predicted by modern portfolio theory.

Our study considers individuals' portfolio choices instead of recommendations from financial advisors. From data on the portfolio composition of 470 clients of a brokerage firm, we have presented a careful verification of the reliability of the risk-tolerance measurement used by the authors. Then, we have obtained that the bonds/stocks ratio does decrease in relation to risk tolerance by using individuals' portfolios. This result complements the findings of Canner, Mankiw, and Weil (1997) and Elton, and Gruber (2000). Finally, we have verified the existence of the two-fund separation theorem in the assets data available to the investors in our sample

Keywords: Credit rating transition matrices, Bayesian estimation, default spread, mobility index, credit contagion, investor rationality, asset allocation puzzle, risk tolerance, separation theorem, bonds/stocks ratio.

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Chapter 1

A Bayesian Framework for Explaining the Rate Spread on Corporate Bonds

1.1 Introduction

The implementation of the Basel II accord scheduled to the end of 2007 has increased the academic and professional interest to credit risk modeling. Therefore, many techniques are developed by the industry such as the KMV model based on Merton's (1974) framework, the CreditRisk+ designed by Credit Suisse (1997) and the CreditMetrics model developed in 1997 by JP Morgan. A key input of the latter model is the credit rating transition matrix provided by rating agencies like Moody's or Standard and Poor's.

The credit rating transition matrix is essentially an overall summary of the probabilities for a particular rating to migrate to other ratings (including default) within a period of time. It should capture the true dynamics of corporate bond credit quality. A change in the rating reflects the assessment that the company's credit quality has improved (upgrade) or deteriorated (downgrade). Thus, an accurate estimation of the transition probabilities is useful for practitioners. It allows, for example, to measure the Credit Value at Risk of bonds portfolio or to price defaultable securities and credit derivatives. For instance, Jarrow, Lando, and Turnbull (1997) have used these transition probabilities to model the term structure of credit risk spreads.

A wider use of the credit rating transition matrix in the future is expected as a consequence of the increasing number of rated firms[1] and the large practice of Internal Ratings Based (IRB) approach. Indeed, the Basel Committee on Banking Supervision (BCBS 2001) recommended the IRB approach since it secures incentive

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[1] The Moody's database documents less than 1300 ratings for the U.S. firms during the eighties, versus more than 2500 ratings since 1997.

compatibility and additional risk sensitivity for the banks: two key objectives of the New Basel Capital Accord. Moreover, Treacy and Carey (2000) document that the internal credit ratings were adopted increasingly by banks during the last two decades. Thus, the estimated transition probabilities should be more accurate and practitioners should be more convinced to utilize such risk management tool.

Many techniques are already available to estimate the transition probabilities. The cohort and the generator methods, described in details in the following section, are widely analyzed by previous studies. The first is attractive by its simplicity while the second is useful to estimate rare migration probabilities. Indeed, highly rated firms do not default in the short term and a nil default probability will be obtained by applying the cohort technique. However, the Basel Committee on Banking Supervision (BCBS 2005) urges banks to consider a minimal one-year probability of default standing at 0.03% when calculating their capital requirements.

In the present study, we propose a new estimation technique based on Bayes theorem. Such technique allows for migration probabilities strictly higher than zero even for non observed events. A second benefit of the Bayesian technique consists on the possibility to derive a statistical distribution of each element of the credit rating transition matrix. Therefore, the confidence sets, the raw and central moments are easily obtained.

The transition probabilities, obtained via the Bayesian technique, will be the key inputs to evaluate the corporate spread explained by the default risk. To do so, we will consider the discrete time model provided by Dionne, Gauthier, Hammami, Maurice, and Simonato (2005) to assess the default spread as a function of the recovery rate and the default probability of a corporate bond.

The remainder of our paper is structured as follows: the next section reviews the usual techniques of estimation of the credit rating transition matrix. The third section describes our Bayesian technique: we document and motivate our choice of the prior distribution and the likelihood function and we expose the obtained posterior distribution. The data description and the estimation results are available in section 1.4. The fifth section relates to the default spreads for corporate bonds measured by use of Dionne et al. (2005) model and the default probabilities derived from the Bayesian technique. Finally, concluding remarks will follow in section 1.6.

1.2 Literature review

Transition matrices are at the center of modern credit risk management. The reports on rating migrations published by Standard and Poor's and Moody's are studied by credit risk managers around the world and several of the most prominent risk management tools, such as JP. Morgan's CreditMetrics and McKinsey's Credit Portfolio View are built on estimates of rating migration probabilities.

Two main techniques are already cited by previous studies to estimate the transition probabilities. We cite the classical cohort method, and the generator method. A brief description of these techniques, their assumptions and limits are now presented:

1.2.1 The cohort method

The transition probabilities reported by the rating agencies are generally computed by use of the cohort technique. The key input of such method is the total rating migrations observed during a specific period and summarized in the matrix N below.

$$N = \begin{bmatrix} n_{11} & \cdots & n_{1K} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & n_{KK} \end{bmatrix} \tag{1.1}$$

where n_{ij} represents the total number of firms rated at the i-th class of risk at the beginning of the period and rated at the j-th class at the end of the same period. The total number of risk classes, including default, stands at K.

Given these migrations, we can estimate the rating transition probability from class *i* to class *j* (denoted by $\hat{p}_{ij}$) as follows:

$$\hat{p}_{ij} = \frac{n_{ij}}{\sum_{j=1}^{K} n_{ij}} \tag{1.2}$$

Finally, the whole estimated rating transition probabilities are summed up in the following matrix $\hat{P}_{cohort}$

$$\hat{P}_{cohort} = \begin{bmatrix} \hat{p}_{11} & \cdots & \hat{p}_{1K} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} \tag{1.3}$$

The major limit of this cohort technique consists on the estimation of rare events. If the migration from class *i* to class *j* does not occur during the specified period, the estimated transition probability $\hat{p}_{ij}$ will be equal to zero. However, such statement is not acceptable for regulatory and logical reasons. First, we recall the minimal one-year probability of default standing at 0.03% recommended by the Basel Committee on Banking Supervision when calculating the capital requirements for banks. Secondly, if during a specific period there are no transitions from class *i* to class *k*, but there are transitions from class *i* to class *j* and from class *j* to class *k* (but by other firms), then the estimated $\hat{p}_{ik}$ should be non-zero, since there is a chance of successive transitions, even if this event does not happen for one single firm in the sample.

1.2.2 The generator method

Lando and Skødeberg (2002) provide a new approach to deal with the estimation of rare events. By use of a continuous time data, one should estimate, as a first step, the following generator matrix Λ.

$$\Lambda = \begin{bmatrix} \hat{\lambda}_{11} & \cdots & \hat{\lambda}_{1K} \\ \vdots & \ddots & \vdots \\ \hat{\lambda}_{K1} & \cdots & \hat{\lambda}_{KK} \end{bmatrix} \tag{1.4}$$

with

$$\begin{cases} \hat{\lambda}_{ij} = \dfrac{m_{ij}(T)}{\int_0^T Y_i(s)ds} & \text{for } i \neq j \\ \hat{\lambda}_{ii} = -\sum_{i \neq j} \hat{\lambda}_{ij} \end{cases} \tag{1.5}$$

where $Y_i(s)$ corresponds to the total number of firms in the rating class i during the time s, and $m_{ij}(T)$ relates to the number of migrations from the rating class i to the rating class j observed over the period T. Therefore, any period of time spent by a firm in the class i will be detected through the denominator

The second and final step to estimate the (K * K) transition probabilities in a time period t consists on computing the matrix exponential of Λ multiplied by t. In other words,

$$\hat{P}_{generator} = \exp(\Lambda t) \tag{1.6}$$

where

$$\exp(\Lambda t) = \sum_{k=0}^{\infty} \frac{(\Lambda t)^k}{k!} \tag{1.7}$$

Applying this generator method allows generally for strictly positive transition probabilities estimates even for rare or non observable events. However, two limits prevent us using such a technique in practice. In general, the continuous time data is not provided by ratings agencies. Moreover, the high stability of particular issues (namely the sovereign compared to the private issues)[2] is problematic: the estimated transition probability for some rare events could be equal to zero even by using continuous time data and the generator method.

1.3 Bayesian estimation of the transition probabilities

In the present section, we will develop our methodology, based on Bayes theorem, to estimate the transition probabilities. Such methodology is attractive for practitioners since it prevents nil estimates of the all the migration probabilities even for highly stable issues and discrete time data. Moreover, it could generate, analytically, a statistical distribution for each element of the transition matrix. Thus, descriptive statistics and confidence sets of each transition probability are easily established.

The obtained statistical distribution, known as the posterior distribution, needs as key inputs a prior distribution and the likelihood function. Both inputs are now described.

1.3.1 Prior distribution

The prior distribution represents our prior information about the parameters that will be estimated. Such prior information could be provided by previous studies, or by expert opinion. As an example, suppose a parameter θ that cannot be negative theoretically (number of defaults per year for example). The chosen prior distribution for θ, such as the Chi square or the Poisson, should be one that can incorporate the available information on the range of θ. In the opposite, the normal distribution would be an inadequate choice since it allows negative values for the parameter.

Besides, the choice of the prior information will induce the posterior distribution of the parameter: different priors will lead to different estimates. To avoid such dependence on priors, one may choose vague or diffuse prior information[3].

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[2] See Standard & Poor's (2007) report for comparative statistics.

[3] Such prior could be found using Jeffrey's rule or Maximal Data Information Prior (known as MDIP) criterion.

In order to select a convenient prior distribution, we should take into account the two fundamental properties of the transition probabilities. The first property concerns the unitary sum of transition probabilities belonging to the same row. The second property relates to the boundaries of the transition probabilities: each migration probability must belong to the [0 , 1] interval. Analytically, we should have:

$$\begin{cases} \sum_{j=1}^{K} p_{ij}^{b} = 1 \text{ for } i \in \{1,\ldots,K\} \\ 0 \le p_{ij}^{b} \le 1 \end{cases} \tag{1.8}$$

where p_{ij}^{b} relates to the migration probability from class i to class j by use of the Bayesian technique.

The Dirichlet distribution looks as a promising candidate to model our prior information. Such multivariate distribution conforms both fundamentals properties discussed above. Therefore, each row $\left(p_{i1}^{b},\ldots,p_{iK}^{b}\right)$ of the transition matrix follows a Dirichlet distribution with K parameters $\left(\alpha_{i1},\ldots,\alpha_{iK}\right)$. In other words, we have:

$$\left(p_{i1}^{b},\ldots,p_{iK}^{b}\right) \rightarrow Dir\left(\alpha_{i1},\ldots,\alpha_{iK}\right) \tag{1.9}$$

with $\alpha_{ij} \ge 0$ for $j \in \{1,\ldots,K\}$. Then, we can derive the distribution function:

$$f\left(p_{i1}^{b},\ldots,p_{iK}^{b}\right) = \frac{\Gamma\left(\sum_{j=1}^{K} \alpha_{ij}\right)}{\prod_{j=1}^{K} \Gamma\left(\alpha_{ij}\right)} \prod_{j=1}^{K} \left(p_{ij}^{b}\right)^{\alpha_{ij}-1} \tag{1.10}$$

where Γ(.) designates the Gamma function: $\Gamma\left(\alpha_{ij}\right) = \int_{0}^{+\infty} t^{\alpha_{ij}-1} e^{-t} dt$

Moreover, the Dirichlet distribution is highly flexible. The variation of its parameters implies different features. Let's examine the 3 dimensional case (K = 3). As shown in Figure 1.1 below, changing each of the 3 parameters $(\alpha_1, \alpha_2, \alpha_3)$ leads to a new shape of the distribution. The top left case $(\alpha_1 = \alpha_2 = \alpha_3 = 1)$ looks more suitable to represent a diffuse prior whereas the bottom right case $(\alpha_1 = \alpha_2 = \alpha_3 = 10)$ corresponds to the prior information with the highest accuracy (among the 4 cases considered).

(Figure 1.1 about here)

1.3.2 Likelihood function

The second input of Bayesian estimation is the likelihood function. The choice of the likelihood function depends on the mechanics of the problem to hand. It is the same problem faced using classical inference: which model should one choose for the available data?

Often, knowledge of the structure by which the data is obtained may suggest appropriate models such as Binomial sampling or Poisson counts. For our case, the observed migrations can be captured by a Multinomial distribution. The N_i firms belonging to the i-th rating class at the beginning of the period will migrate to one of the K rating classes. Thus, we have the equality:

$$N_i = \sum_{j=1}^{K} n_{ij} \tag{1.11}$$

where n_{ij} designates the total number of firms rated at the i-th class of risk at the beginning of the period and rated at the j-th class at the end of the same period.

Therefore, the Multinomial distribution of the N_i firms is derived conditional on the parameters p_{ij}^b defined earlier. In other words,

$$\Pr\left(n_{i1},\ldots,n_{iK}\middle|p_{i1}^{b},\ldots,p_{iK}^{b}\right)=\frac{N_i!}{\prod_{j=1}^{K}n_{ij}!}\prod_{j=1}^{K}\left(p_{ij}^{b}\right)^{n_{ij}} \qquad (1.12)$$

1.3.3 Posterior distribution

Once we defined the prior distribution and the likelihood function, we can derive the posterior distribution by use of the Bayes Theorem. The distribution function of the estimated probabilities $\left(p_{i1}^{b},\ldots,p_{iK}^{b}\right)$, conditional on the observed migrations $\left(n_{i1},\ldots,n_{iK}\right)$ is the following:

$$\Pr\left(p_{i1}^{b},\ldots,p_{iK}^{b}\middle|n_{i1},\ldots,n_{iK}\right)\propto f\left(p_{i1}^{b},\ldots,p_{iK}^{b}\right)*\Pr\left(n_{i1},\ldots,n_{iK}\middle|p_{i1}^{b},\ldots,p_{iK}^{b}\right) \qquad (1.13)$$

Deriving this distribution function leads us to:

$$\Pr\left(p_{i1}^{b},\ldots,p_{iK}^{b}\middle|n_{i1},\ldots,n_{iK}\right)\propto\prod_{j=1}^{K}\left(p_{ij}^{b}\right)^{n_{ij}+\alpha_{ij}-1} \qquad (1.14)$$

which corresponds to the kernel of a Dirichlet distribution with K parameters, namely $\left(n_{i1}+\alpha_{i1},\ldots,n_{iK}+\alpha_{iK}\right)$. Thus the Dirichlet distribution corresponds to a conjugate prior which is a useful feature: it allows for updating estimated results by adding new data.

Thus, the Bayesian technique allows us to specify the whole distribution of each row of the transition matrix. However, for practical reasons, we should report only some statistics of the derived distributions. For example, the mean of the estimated probability of p_{ij}^{b} equals:

$$E\left(p_{ij}^{b}\right)=\frac{n_{ij}+\alpha_{ij}}{\sum_{j=1}^{K}\left(n_{ij}+\alpha_{ij}\right)}=\frac{n_{ij}+\alpha_{ij}}{N_i+\sum_{j=1}^{K}\left(\alpha_{ij}\right)} \qquad (1.15)$$

Focusing on this point estimate allows us to conclude with three main remarks. First, the mean estimated migration probability is an increasing function with reference to n_{ij}. Secondly, it decreases with reference to N_i: the impact of the priors will be minimized when considering large samples. This is not be surprising since larger samples are synonymous of higher accuracy. Finally, by considering more classes (increasing *K*) we diminish all point estimates.

The variance of each estimated transition probability equals:

$$V\left(p_{ij}^{b}\right)=\frac{\left(n_{ij}+\alpha_{ij}\right)\left[\left[\sum_{j=1}^{K}\left(n_{ij}+\alpha_{ij}\right)\right]-\left(n_{ij}+\alpha_{ij}\right)\right]}{\left[\sum_{j=1}^{K}\left(n_{ij}+\alpha_{ij}\right)\right]^{2}\left[\left[\sum_{j=1}^{K}\left(n_{ij}+\alpha_{ij}\right)\right]+1\right]} \tag{1.16}$$

1.4 Empirical results

Once we have exposed our Bayesian framework to estimate the transition probabilities, we will compare the results obtained if applying such technique with those induced by the cohort method. To do so, we start by describing the data used. The comparative results follow.

1.4.1 Data description

Our database consists on the yearly transition matrices available from Moody's, from 1987 to 1996 for the U.S. industrial bonds. Thus, we take into account only the rating of each firm at the beginning and at the end of each year. The withdrawn issues are discarded from the analysis since they cannot be explained by a deterioration of the credit quality. Finally, we consider 8 classes of risk, namely Aaa, Aa, A, Baa, Ba, B, Caa-C and default. Such classification is privileged by previous studies to deal with the data scarcity. The total migrations observed during the whole period are summarized in the following table.

(Table 1.1 about here)

From Table 1.1, we notice that all bonds rated Aaa, Aa or A at the beginning of the year did not default by the end of the same year. Thus, by applying the cohort estimation, the default probability for each of these bonds equals zero. The other estimated transition probabilities obtained by the cohort or Bayesian techniques will be discussed in the next subsections.

1.4.2 Cohort transition probabilities

Based on migrations shown in Table 1.1, we obtain the following cohort transition probabilities

(Table 1.2 about here)

We remark the high stability of the ratings during a year, especially for investment grade issues. More than 88% of these issues keep the same rating within a year, whereas only 66% of the Caa-C rated issues remain in the same class within the same period.

1.4.3 Bayesian transition probabilities

The Bayesian estimation needs the determination of the prior distribution parameters. In other words, we have to make assumptions about the α_{ij} terms defined in (1.9). For instance, we assume a pre-defined structure of these terms that captures some stylized facts. For our case, we suppose that:

$$\alpha_{ij} = \theta^{|i-j|} \quad \forall\, j \in \{1,\ldots,K\} \text{ and } i \in \{1,\ldots,K-1\} \tag{1.17}$$

with $\theta \leq 1$.

Thus, knowing the scalar θ allows us to derive the $K(K - 1)$ terms α_{ij}. Moreover, with such a structure, we give more emphasis to the diagonal elements of the transition matrix. As cited by previous studies, the highest transition probabilities correspond usually to the non migration cases. The transition probabilities should decrease for the off-diagonal elements: the far we are from the diagonal, the lower is the transition probability. The structure defined above takes into account the gap between the initial and the final ratings to derive the α_{ij}.

A particular discussion should be accorded to the ($\theta = 1$) case. For such a situation, we give the same importance for all transition probabilities. In other words, we consider the least informative prior, i.e. the prior that provides no additional information when estimating the transition probabilities.

However, a θ strictly lower than one looks more realistic. Indeed, we notice generally that more than 50% of the issues remain in the same rating within the year[4]. Also, minor rating changes (for example from Aaa to Aa) are generally more frequent than higher rating changes (from Aaa to B for example). We can take into account such information by choosing a θ lower than one.

By setting a θ equal to (1/4), we obtain a prior information that conforms some stylized facts: the proportion of issues keeping the same rating within a year is higher than 60% and decreasing transition probabilities are observed for higher gaps between the initial and final ratings. Thus, by applying the expression (1.15) to our database, we obtain the following average transition probabilities by use of the Bayesian estimation[5].

(Table 1.3 about here)

We notice the usefulness of the Bayesian technique: it allows for non zero transition probabilities, on average, even for the non observed transitions such as the migration from the Aaa to default. Also, we remark the monotonicity property: a lower credit

[4] Such feature is also observable from our database. More than 66% of the issues had kept the same rating within the year.
[5] Others values of θ will be considered later to assess the robustness of our results.

quality is synonymous of higher estimated default probability. Moreover, we recall the possibility to derive analytically the standard deviation of each transition probability. Following the result in (1.16), the standard deviations of the estimated transition probabilities are equal to:

(Table 1.4 about here)

From Table 1.4, we notice higher levels for the standard deviations of the rating classes with fewer observations at the beginning of the year. For example, the total number of A and Caa-C rated issues at the beginning of the year stands, respectively, at 5205 and 256. The average standard deviation for the first rating class equals 0.132% while the same average for the second class equals 1.100%.

Furthermore, we can draw the statistical distribution of each Bayesian transition probability. The following plots display the statistical distribution of some of them[6]:

(Figure 1.2 about here)

From Figure 1.2, we notice the skewness of some statistical distributions: the estimated transition probability from Baa to default is right skewed whereas the estimated probability to remain in Aaa class looks left skewed.

Also, the Bayesian technique allows us to derive the confidence sets of each transition probability, as done by Christensen, Hansen, and Lando (2004). Simulating the obtained posterior distributions (a Dirichlet distribution for each rating class), allows us to derive the following 99.9% confidence intervals for the Bayesian transition probabilities.

(Table 1.5 about here)

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[6] These univariate statistical distributions are obtained via 100 000 simulations from the Dirichlet distribution.

Again, we notice the monotonicity property from Table 1.5. The lower and upper bounds of estimated default probabilities never increase for higher credit quality issues.

Finally, it is worth noting that all the previous results relating to the Bayesian estimation are derived by assuming a θ equal to (1/4). Thus, we should check the robustness of our results by considering different values of the θ parameter. To do so, we simulate the Bayesian transition probabilities by applying expressions (1.14) and (1.17) for various values of θ. We consider a θ equal to one (synonymous to a diffuse prior) and (1/2). Table 1.6 below reports the mean and 99.9% confidence intervals for one year default probabilities by assuming different values of θ.

(Table 1.6 about here)

Four main remarks should be noticed from Table 1.6. First, we observe the superiority of the Bayesian technique, with comparison to the cohort technique: the average default probability is strictly higher than zero for each issue and each θ considered. Secondly, the confidence intervals obtained with $\theta = 1/2$ or $\theta = 1$ for Aaa, Aa, A rated issues, include the minimal one year default probability (0.03%) set by the Basel Committee on Banking Supervision (BCBS 2005). A nil one year default probability would be obtained for the same issues if applying the cohort technique.

The third remark concerns the monotonicity property. For instance, upper bounds of the Aaa default probabilities are higher then those of Aa rated issues with $\theta = 1/2$ or $\theta = 1$. Such result should not be considered as counterintuitive since it indicates only the boundaries of confidence intervals. We expect wider confidence intervals for Aaa rated issues (with comparison to Aa rated issues) as a consequence of their lower sample size[7] and consequently their higher standard deviation derived from (1.16)[8].

[7] Following Table 1.1, the total number of Aaa and Aa rated issues stands at 612 and 2050 respectively.
[8] For $\theta = 1$ (*respec.* $\theta = 1/2$), the standard deviation of the Aaa probability of default equals 0.161% (*respec.* 0.014%) whereas it stands at 0.049% (*respec*. 0.006%) for the Aa probability of default.

Let's focus on the $\theta = 1$ case: the upper bound of the Aa default probability is larger than the lower bound of Aaa default probability. Thus, we cannot confirm that the Aa default probability is lower then the Aaa default probability. By same reasoning, we cannot reject the monotonicity property for the whole Bayesian default probabilities displayed in Table 1.6.

Finally, we notice the impact of a selected prior distribution on the estimated default probabilities. A higher θ increases the default probabilities especially for issues never defaulting during the observed period (namely Aaa, Aa and A rated issues). Thus, by increasing θ, we accord relatively more confidence to the expert opinion (with reference to the information provided from the database).

1.5 Default spreads

In order to assess the impact of our Bayesian estimates on risk management, we will compare the default spreads[9] obtained by use of the cohort method and those of the Bayesian technique. To do so, we will consider the discrete time framework of Dionne et al.(2005) to derive the default spreads from the default probabilities.

Following Dionne et al.(2005), the default spread at time t of a defaultable bond, maturating in $(T - t)$ periods, equals:

$$S(t,T) = -\frac{\ln p_{T-t}}{\Delta_t (T-t)} \tag{1.18}$$

with

$$\begin{cases} p_s = \rho \sum_{u=1}^{s} p_{s-u} q_u + \left(1 - \sum_{u=1}^{s} q_u\right) \text{ for } s \in \{1, 2, \ldots\} \\ p_0 = 1 \end{cases} \tag{1.19}$$

[9] The corporate rate spreads which can be attributed to default risk.

where q_u designates the probability, under the risk-neutral measure Q, that the default will occur in exactly u periods from now and ρ corresponds to the recovery rate.

For the present study, we will focus on the default spread of the investment grade bonds, namely the issues rated Aaa, Aa, A or Baa with maturities ranging from one to ten years. By analogy to Elton, Gruber, Agrawal, and Mann (2001), we will consider the recoveries rates reported by Altman and Kishore (1998). In other words, the recovery rates for Aaa, Aa, A and Baa issues stand at 68.34%, 59.59%, 60.63% and 49.42%, respectively.

Let's start by computing the cohort and Bayesian default probabilities. A θ equal to (1/4) is considered to derive the Bayesian probabilities. Table 1.7, below, reports the cumulative default probabilities for investment grade issues for maturities ranging from one to ten years.

(Table 1.7 about here)

From Table 1.7, we detect the impact of our Bayesian estimates especially for Aaa rated issues. The cumulative default probabilities of Aaa issues obtained by both techniques (cohort and Bayesian) roughly converge for maturities higher than five years. The cumulative default probabilities convergence is faster for the remaining issues.

Recall that the adjustment recommended by the Basel Committee on Banking Supervision relates only to short maturities. The convergence noticed in Table 1.7 is in favour of our Bayesian specification: for short maturities, we obtain non zero default probabilities, even for highly rated issues. The default probabilities for longer maturities are similar to those obtained by cohort technique.

We notice that the default probabilities used in (1.19) to measure the default spreads correspond to the marginal default probabilities and not to the cumulative probabilities reported in Table 1.7. Recall that the marginal probability of default during year u indicates the conditional probability of defaulting during year u, knowing that no default occurs before. Thus, denoting by τ the year of default, the marginal probability of default during year u is the following:

$$P(\tau = u / \tau > u-1) = \frac{P(\tau = u; \tau > u-1)}{P(\tau > u-1)} = \frac{P(\tau = u) - P(\tau = u-1)}{1 - P(\tau = u-1)} \qquad (1.20)$$

where $P(\tau = u)$ denotes the cumulative default probability during year u.

By applying (1.18) and (1.19), we obtain the following default spreads for investment grade issues and maturities ranging from one to ten years.

(Table 1.8 about here)

Again, we observe the similarity of cohort and Bayesian techniques for long maturities. For shorter maturities, the corporate spreads explained by default risk is higher by applying the Bayesian technique.

1.6 Conclusions

In this paper we presented an alternative approach for estimating the one-year transition matrix based on a Bayesian framework. The results reveal that the default probabilities resulting from this study are non-zero for the highest ratings unlike what is reported in Moody's and Standard and Poor's transition matrices. Moreover, this Bayesian technique allows deriving confidence intervals for the transition probabilities and, thus, conforming the regulatory concerns about out of sample testing

We then focused on the estimation of the default spread using the Bayesian transition matrix. The results obtained show that the corporate spreads explained by default risk are higher for short maturities and high rated issues, relatively to those computed from the cohort transition matrix. Both techniques lead roughly to the same default spreads for longer maturities.

As an extension, we should test if the results obtained are specific to the likelihood function and the prior distribution used for the Bayesian estimation. Simulation methods could be used if the posterior distribution cannot be expressed analytically.

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Table 1.1: Total yearly migrations for the U.S. industrial bonds from 1987 to 1996

Table 1.1 reports rating migrations for the U.S. industrials bonds from 1987 to 1996. 8 classes of credit risk are considered, including the default state.

	Aaa	Aa	A	Baa	Ba	B	Caa-C	D
Aaa	570	41	1	0	0	0	0	0
Aa	16	1823	205	3	2	1	0	0
A	2	86	4819	255	34	8	1	0
Baa	2	86	213	3061	156	26	3	3
Ba	1	11	22	168	3055	315	12	71
B	1	1	8	22	181	2411	84	240
Caa-C	0	0	0	2	5	25	169	55

Table 1.2: Cohort transition probabilities for the U.S. industrial bonds (%)

Following Table 1.1 and Equation (1.2), we obtain these transition probabilities for the 8 classes of credit risk.

	Aaa	Aa	A	Baa	Ba	B	Caa-C	D
Aaa	93.14	6.70	0.16	0.00	0.00	0.00	0.00	0.00
Aa	0.78	88.93	10.00	0.15	0.10	0.05	0.00	0.00
A	0.04	1.65	92.58	4.90	0.65	0.15	0.02	0.00
Baa	0.06	0.32	6.13	88.09	4.49	0.75	0.09	0.09
Ba	0.03	0.03	0.60	4.61	83.81	8.64	0.33	1.95
B	0.03	0.10	0.27	0.75	6.14	81.73	2.85	8.14
Caa-C	0.00	0.00	0.00	0.78	1.95	9.77	66.02	21.48

Table 1.3: Mean Bayesian transition probabilities (%)

Considering a θ = (1/4) and following Table 1.1 and Equation (1.15), we obtain these mean Bayesian transition probabilities for the 8 classes of risk.

	Aaa	Aa	A	Baa	Ba	B	Caa-C	D
Aaa	93.10	6.73	0.17	2.6e-03	6.4e-04	1.6e-04	4.0e-05	1.0e-05
Aa	0.79	88.91	10.00	0.15	0.10	0.05	4.8e-05	1.2e-05
A	0.04	1.66	92.57	4.90	0.65	0.15	0.02	1.9e-05
Baa	0.06	0.32	6.13	88.07	4.49	0.75	0.09	0.09
Ba	0.03	0.03	0.61	4.61	83.80	8.64	0.33	1.95
B	0.03	0.10	0.27	0.75	6.14	81.72	2.85	8.13
Caa-C	9.5e-05	3.8e-04	1.5e-03	0.78	1.97	9.80	66.00	21.45

Table 1.4: Standard deviation of Bayesian transition probabilities (%)

Table 1.4 reports the standard deviation of Bayesian transition probabilities obtained by considering a θ = (1/4) and following Table 1.1 and Equation (1.16).

	Aaa	Aa	A	Baa	Ba	B	Caa-C	D
Aaa	1.023	1.011	0.168	0.020	0.010	0.005	0.003	0.001
Aa	0.196	0.693	0.662	0.085	0.069	0.049	0.002	7.6e-04
A	0.028	0.177	0.363	0.299	0.112	0.054	0.019	6.0e-04
Baa	0.041	0.096	0.407	0.550	0.351	0.146	0.05	0.050
Ba	0.027	0.028	0.128	0.347	0.610	0.465	0.095	0.229
B	0.034	0.059	0.096	0.159	0.442	0.711	0.306	0.503
Caa-C	0.006	0.012	0.024	0.548	0.863	1.849	2.946	2.553

Table 1.5: Confidence intervals of Bayesian transition probabilities (%)

Table 1.5 displays the 99.9% confidence interval of each Bayesian transition probability. 100 000 simulations from the Dirichlet distribution are used, by considering a $\theta = (1/4)$. Panel A reports the lower bounds of each confidence interval. Panel B reports the upper bounds.

Panel A: Lower bounds of Bayesian transition probabilities (%)

	Aaa	Aa	A	Baa	Ba	B	Caa-C	D
Aaa	89.31	3.84	1.7e-04	0.00	0.00	0.00	0.00	0.00
Aa	0.29	86.49	7.94	0.01	1.9e-03	2.3e-05	0.00	0.00
A	6.3e-04	1.14	91.33	3.98	0.35	0.03	1.1e-05	0.00
Baa	9.9e-04	0.09	4.89	86.21	3.44	0.36	4.3e-03	4.1e-03
Ba	1.8e-05	1.4e-05	0.27	3.52	81.70	7.21	0.10	1.29
B	1.8e-05	4.9e-03	0.06	0.33	4.78	79.29	1.94	6.58
Caa-C	0.00	0.00	0.00	0.01	0.24	4.77	55.91	13.94

Panel B: Upper bounds of Bayesian transition probabilities (%)

	Aaa	Aa	A	Baa	Ba	B	Caa-C	D
Aaa	96.02	10.43	1.26	0.37	0.24	0.10	0.01	2.6e-06
Aa	1.59	91.06	12.31	0.59	0.49	0.37	0.03	6.2e-03
A	0.19	2.28	93.73	5.95	1.08	0.40	0.15	8.0e-03
Baa	0.29	0.72	7.53	89.80	5.75	1.33	0.34	0.35
Ba	0.21	0.21	1.11	5.83	85.76	10.25	0.73	2.79
B	0.27	0.41	0.70	1.40	7.68	84.00	3.96	9.89
Caa-C	0.02	0.20	0.50	3.84	6.13	16.75	75.07	30.57

Table 1.6: Default probabilities for various prior's structures (%)

Panel A of Table 1.6 reports the mean Bayesian default probabilities for various values of parameter θ. Panel B reports the 99.9% confidence interval of each Bayesian default probability via 100 000 simulations from the Dirichlet distribution.

Panel A: Mean Bayesian transition probabilities (%)

	Cohort	$\theta = 1/4$	$\theta = 1/2$	$\theta = 1$
Aaa	0.00	1.0e-05	0.001	0.16
Aa	0.00	1.2e-05	0.001	0.05
A	0.00	1.9e-05	0.001	0.02
Baa	0.09	0.09	0.09	0.12
Ba	1.95	1.95	1.95	1.97
B	8.14	8.13	8.14	8.15
Caa-C	21.48	21.45	21.47	21.21

Panel B: Confidence intervals of Bayesian transition probabilities (%)

	Cohort	$\theta = 1/4$	$\theta = 1/2$	$\theta = 1$
Aaa	0.00	[0.00 ; 2.6e-06]	[0.00 ; 0.30]	[7.4e-05 ; 1.20]
Aa	0.00	[0.00 ; 6.2e-03]	[0.00 ; 0.11]	[2.7e-05 ; 0.36]
A	0.00	[0.00 ; 8.0e-03]	[0.00 ; 0.05]	[9.9e-06 ; 0.14]
Baa	0.09	[4.1e-03 ; 0.35]	[4.5e-03 ; 0.35]	[0.01 ; 0.40]
Ba	1.95	[1.29 ; 2.79]	[1.28 ; 2.80]	[1.30 ; 2.83]
B	8.14	[6.58 ; 9.89]	[6.58 ; 9.88]	[6.56 ; 9.92]
Caa-C	21.48	[13.94 ; 30.57]	[13.91 ; 30.65]	[13.74 ; 29.99]

Table 1.7: Cumulative default probabilities (%)

Table 1.7 displays the cumulative default probabilities for investment grade bonds with maturities ranging from 1 to 10 years, by applying the cohort and Bayesian techniques. A $\theta = (1/4)$ is considered for the Bayesian technique.

Years	Cohort				Bayesian			
	Aaa	Aa	A	Baa	Aaa	Aa	A	Baa
1	0.00	0.00	0.00	0.09	1.0e-05	1.2e-05	1.9e-05	0.09
2	0.00	6.0e-03	0.03	0.33	5.6e-05	6.1e-03	0.03	0.33
3	4.6e-04	0.02	0.11	0.72	6.3e-04	0.02	0.11	0.72
4	2.0e-03	0.05	0.22	1.25	2.3e-03	0.05	0.22	1.25
5	5.5e-03	0.09	0.39	1.91	6.0e-03	0.09	0.39	1.91
6	0.01	0.15	0.60	2.68	0.01	0.15	0.60	2.68
7	0.02	0.23	0.87	3.54	0.02	0.23	0.87	3.54
8	0.04	0.34	1.18	4.48	0.04	0.34	1.18	4.48
9	0.06	0.47	1.54	5.49	0.06	0.47	1.54	5.49
10	0.09	0.63	1.96	6.55	0.09	0.63	1.96	6.55

Table 1.8: Default spreads (%)

Table 1.8 displays the default spreads (corporate rate spreads which can be attributed to default risk) for investment grade bonds with maturities ranging from 1 to 10 years, by applying the cohort and Bayesian techniques. A θ = (1/4) is considered for the Bayesian technique.

Years	Cohort				Bayesian			
	Aaa	Aa	A	Baa	Aaa	Aa	A	Baa
1	0.00	0.00	0.00	0.04	3.2e-06	4.8e-06	7.4e-06	0.04
2	0.00	1.2e-03	6.6e-03	0.08	8.9e-06	1.2e-03	6.6e-03	0.08
3	4.8e-05	2.8e-03	0.01	0.12	6.4e-05	2.9e-03	0.01	0.12
4	1.6e-04	4.9e-03	0.02	0.16	1.8e-04	4.9e-03	0.02	0.16
5	3.5e-04	7.3e-03	0.03	0.20	3.8e-04	7.3e-03	0.03	0.20
6	6.2e-04	0.01	0.04	0.23	6.6e-04	0.01	0.04	0.23
7	1.0e-03	0.01	0.05	0.26	1.0e-03	0.01	0.05	0.26
8	1.5e-03	0.02	0.06	0.29	1.5e-03	0.02	0.06	0.29
9	2.1e-03	0.02	0.07	0.32	2.2e-03	0.02	0.07	0.32
10	2.8e-03	0.03	0.08	0.35	2.9e-03	0.03	0.08	0.35

Figure 1.1: Different features of the Dirichlet distribution

Figure 1.1 displays the flexibility of the Dirichlet distribution. By changing each of the 3 parameters $(\alpha_1, \alpha_2, \alpha_3)$, we obtain a new shape of the distribution. The top left case $(\alpha_1 = \alpha_2 = \alpha_3 = 1)$ looks more suitable to represent a diffuse prior whereas the bottom right case $(\alpha_1 = \alpha_2 = \alpha_3 = 10)$ corresponds to the prior information with the highest accuracy (among the 4 cases considered).

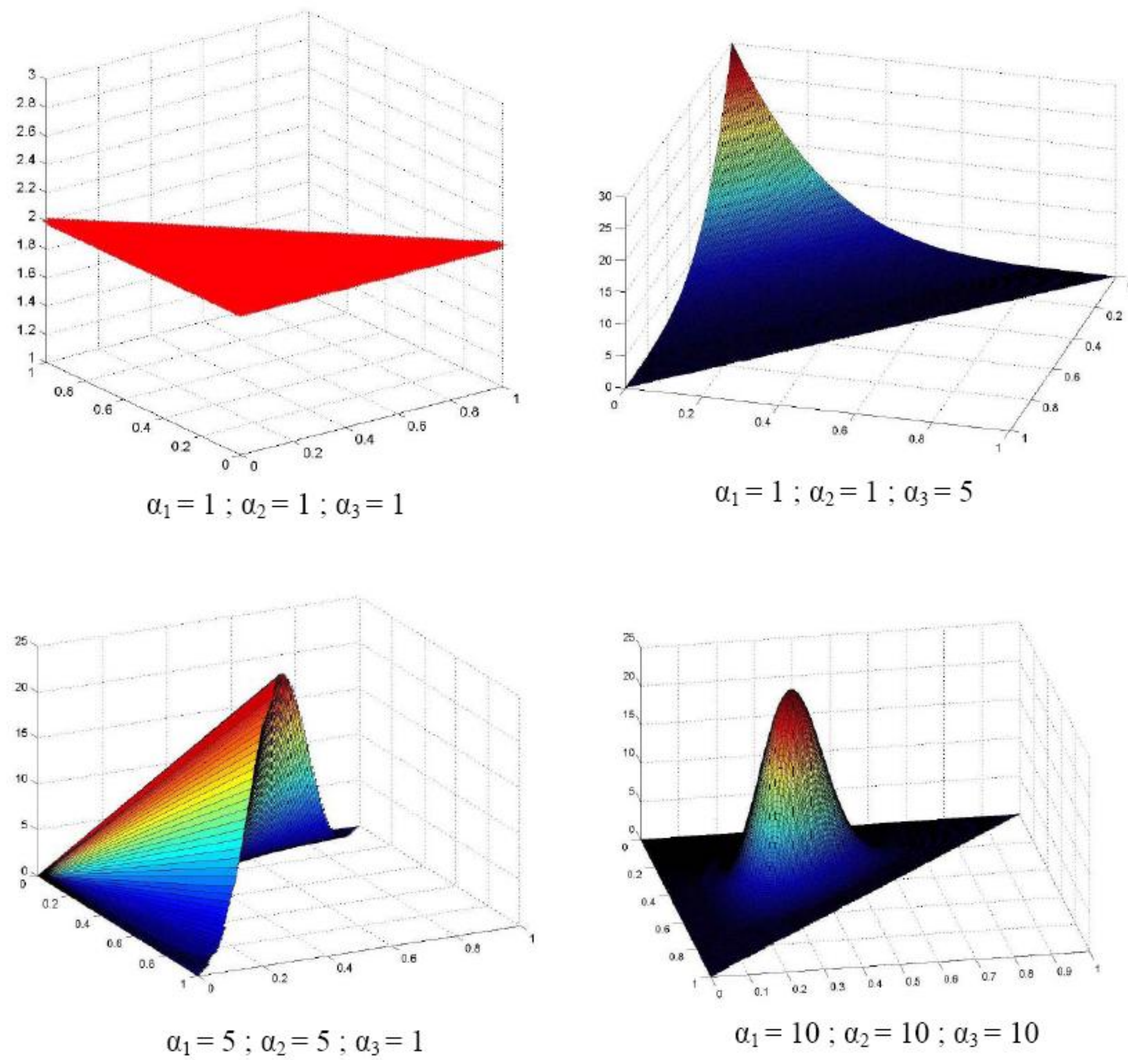
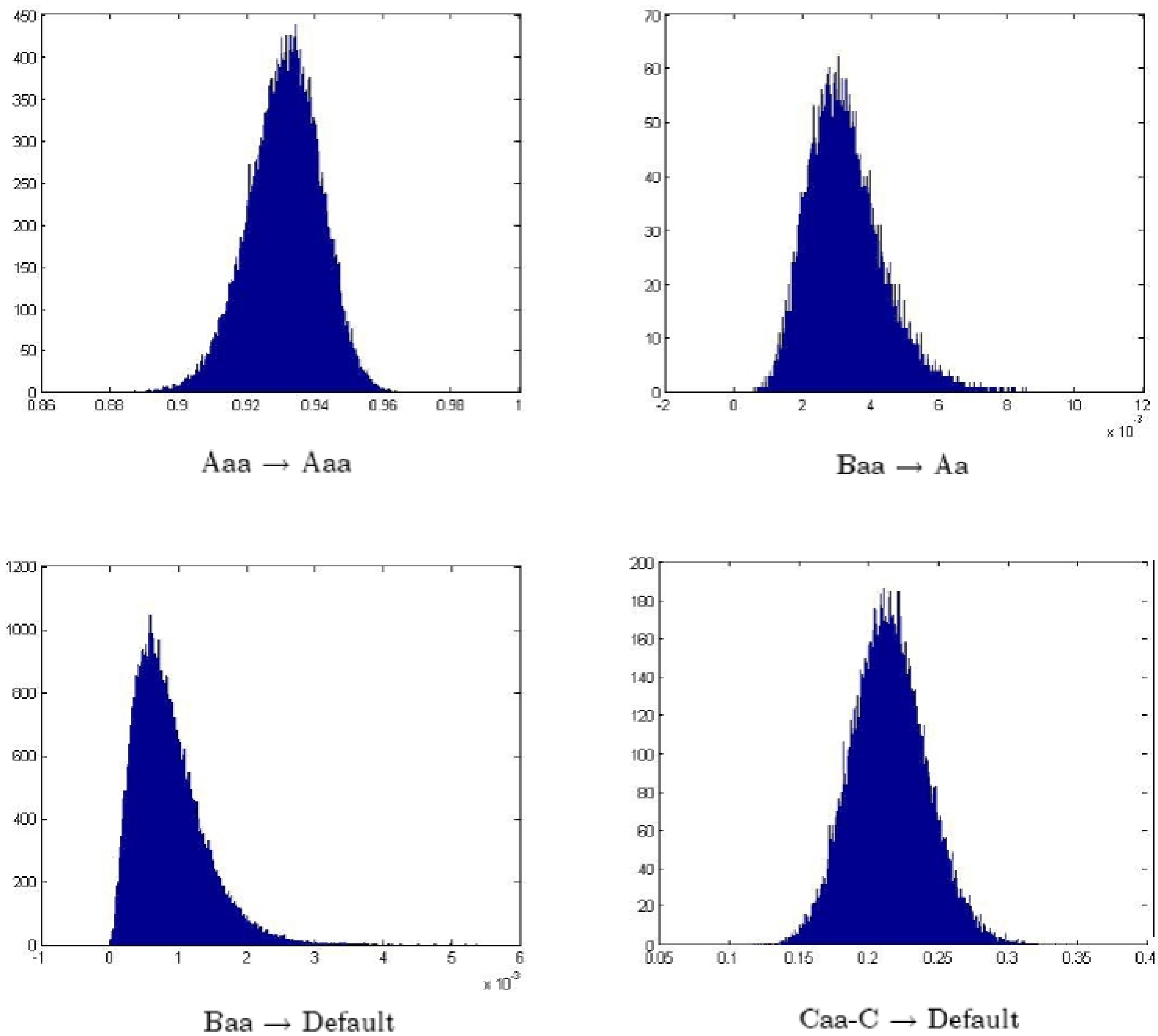


Figure 1.2: Statistical distributions of some Bayesian transition probabilities

Based on 100 000 simulations from the Dirichlet distribution and θ = (1/4), we obtain the following statistical distributions for some specific Bayesian transition probabilities.



Chapter 2

Migration Dependence among the U.S. Business Sectors

2.1 Introduction

The New Basel Accord (BCBS 2004) urges financial institutions to measure and evaluate credit risk by using credit rating matrices. These inputs are generally provided by external credit rating agencies such as Standard & Poor's and Moody's or obtained directly by internal ratings. Both methods are widely used in practice but we can mention the analysis of Treacy and Carey (2000) who documents that these internal credit ratings were adopted increasingly by banks during the last two decades.

Evaluating the total loss on a portfolio resulting from the default of a company's obligors should take into account the migration correlation. An omission or an inaccurate estimation of that correlation will induce a misestimation of the regulatory capital and, consequently, will embrittle the company's financial stability. Moreover, Zhou (2001) and Das, Duffie, Kapadia and Saita (2007) document that default correlation analysis is needed for asset pricing. Such data is fundamental when evaluating a basket default swap or a Collateralized Debt Obligation. Thus, an efficient estimation of the migration correlation is necessary to assess the credit risk.

Gagliardini and Gouriéroux (2005) provide an efficient way to estimate the joint migration probabilities and the migration correlations. However, two major limits are advanced to their methodology. First, they do not consider the contagion effects: the transmission of credit shocks from one firm or sector to the remaining firms. The studies of Giesecke and Weber (2004, 2006) evaluate the credit losses on portfolios by considering both the cross-sectional correlation (a consequence of the common factors affecting all firms) and the credit contagion (caused by the business links between firms). Das et al.(2007) show, empirically, the need to consider contagion effects when modelling default correlation. The second limit concerns the practical aspect: the number of migration correlations to report grows exponentially with the number of credit classes considered. Identifying K classes of credit risk, including the default state which is an absorbent state, will generate $[K(K-1)]^2$ migration correlations. Thus, the 8 classes of credit risk considered by major rating agencies will induce a calculus of 3136 correlations. This dimensionality problem will be amplified once we distinguish between growth and recession cycles or between different business sectors.

In fact, the time variation of default correlation should be considered by practitioners. The Das, Freed, Geng and Kapadia (2006) study documents the presence of two regimes of default correlation when focusing on the U.S. public non-financial firms from 1987 to 2000. Moreover, these authors explain the limits of the CreditMetrics methodology. They argue that "asset return correlations are relatively stable over time" when compared to the default correlation.

Thus, we propose the use of the mobility indices mentioned by Jafry and Schuermann (2004) to test for the existence of correlation migration and contagion effects between business sectors. Moreover, we will check if this dependence is cyclical. The mobility indices allow us to summarize the whole transition matrix by a scalar. We can, then, verify if the entire sector is, on average, in an upgrade or downgrade stage. This recapitulation will circumvent the dimensionality limit cited above. It will permit, also, the disentanglement of credit contagion and the cross-sectional correlation.

In addition, we test if rating changes in a specific business sector affect the next rating changes for the same sector. We held a second test to analyze possible linkages between rating transition probabilities and the business cycle. For example, the study of Nickell, Perraudin and Varotto (2000) reveals that the business cycle affects the rating transition matrix: higher downgrading probabilities are observed during recessions compared to expansion periods.

Based on quarterly rating transitions provided by Moody's for the period January 1980 to April 2005, we evaluate the mobility indices for corporate issuers belonging to various U.S. business sectors. The results obtained confirm the presence of a crisis transmission phenomenon within some business sectors. They also prove the suspected linkage between macro economic factors and transition probabilities. A third result relates to the dependence between the various sectors. Generally, we are able to identify two Markov switching regimes for the migration correlation between two business sectors. Furthermore, the contagion effects vary according to each regime. These shock transmissions are measured by the impulse response functions.

The remainder of our paper is structured as follows: the next section introduces briefly the mobility indices and their applications. The third section describes the data: the transition matrices provided by Moody's and the macro economic variables. We will explain our choice of a specific mobility index in section 2.4. We analyze the obtained mobility indexes for each business sector in the fifth section and we estimate migration dependence among sectors in the sixth section. Finally, concluding remarks will follow in section 2.7.

2.2 Mobility indices

Shorrocks (1978) was the first to introduce the concept of mobility indexes. Such a metric is produced to sum up the dynamic part of a transition matrix (off-diagonal probabilities). Therefore, it maps the whole matrix to a scalar and allows easier comparison among several transition matrices. For example, Jafry and Schuermann (2004) apply these metrics to compare the credit transition matrices obtained by various estimation techniques. Many mobility indices are cited in the literature. They can be classified into three categories.

2.2.1 Cell by cell distance metrics

These metrics are generally obtained via a simple addition of the off-diagonal elements of a transition matrix. Suppose the ($K*K$) transition matrix P and the identity matrix *I(K)*, where *K* corresponds to the total number of credit classes. Also, let x_{ij} the transition probability from class *i* to class *j*.

$$P = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1K} \\ x_{21} & x_{22} & \cdots & x_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ x_{K-1,1} & x_{K-1,2} & & x_{K-1,K} \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

The most intuitive metrics belonging to this category can be defined as following:

$$DC1 = \sum_{i=1}^{K}\sum_{j=1}^{K}\left|\tilde{x}_{ij}\right| \qquad (2.1)$$

$$DC2 = \sum_{i=1}^{K}\sum_{j=1}^{K}\tilde{x}_{ij}^{2} \qquad (2.2)$$

$$DC3 = \sum_{i=1}^{K}\sum_{j=1}^{K}(i-j)\tilde{x}_{ij} \qquad (2.3)$$

where $\tilde{x}_{ij}$ indicates the elements of the mobility matrix $\tilde{P}$ defined in Jafry and Schuermann (2004) as the difference between *P* and *I*.

$$\tilde{P} = P - I \qquad (2.4)$$

2.2.2 Eigenvalue based metrics

Geweke, Marshall and Zarkin (1986) discussed various mobility indices based on the eigenvalues of *P*. Here are few examples of these metrics:

$$DEVA1 = 1 - |\det(P)| \qquad (2.5)$$

$$DEVA2 = 1 - |\lambda_2(P)| \qquad (2.6)$$

$$DEVA3 = \frac{\log(0.5)}{\log(\lambda_2)} \qquad (2.7)$$

where det(...) denotes the *determinant*, and λ_i (...) denotes the *i*-th eigenvalue (sorted from largest to smallest absolute value).

The third metric is known as the half-life measure. It allows the calculus of the time needed for the system to decay to 50% of the steady state (the default state when considering credit rating matrices).

2.2.3 Singular value based metrics

Jafry and Schuermann (2004) devised a new mobility index. They prove that the average of the singular values of the matrix *P* defined in (2.4) gives an idea about the dynamic part of a transition matrix. This new metric is obtained as following:

$$DSV = \frac{\sum_{i=1}^{K} \sqrt{\lambda_i\left(\tilde{P}'\tilde{P}\right)}}{K} \qquad (2.8)$$

Finally, we can mention the metric proposed by Arvanitis, Gregory and Laurent (1999). They suggest that the similarity between two matrices (*P* and *Q* for example) could be detected according to the following ratio:

$$DEVE = \frac{\|PQ - QP\|}{\|P\| * \|Q\|} \qquad (2.9)$$

where $\|\ldots\|$ designates the norm of the matrix.

The ratio defined in (2.9) is delimited between zero and two: a nil ratio indicates that P and Q have the same eigenvectors. However, this eigenvector based metric presents a major limitation: we can only have an idea about the resemblance among two matrices. It is impossible, for example, to assess the mobility magnitude of a given matrix or to consider the identity matrix as a benchmark.

2.3 Data

2.3.1 Transition matrices

The key input of our study is the rating transition matrices provided by Moody's. We focus on the corporate issuers domiciled in the United States between January 1980 and April 2005. By observing quarterly the rating of each issuer, considering only the 8 major categories (Aaa, Aa, ..., Caa-C, default) and excluding the "withdrawn rating", we obtain 208 361 issuer ratings. We notice that the total number of withdrawals stands at 3340. A brief investigation of the database shows that the "withdrawn rating" is not synonymous with high default risk.

(Table 2.1 about here)

As shown in Table 2.1, some Aaa rated issues are withdrawn during the following quarter (as a result of debt expiration, for example). Thus, we will follow Carty's(1997) methodology and exclude the "withdrawn rating" when estimating the migration matrices[1].

Figure 2.1 below indicates that the total number of ratings is increasing over time: less than 1300 ratings per quarter are observed during the eighties, versus more than 2500 ratings per quarter since 1997. This situation is explained by the new issues (6099) exceeding the sum of withdrawals and defaults (standing at 3340 and 1073 respectively).

[1] For more details about withdrawn rating's adjustment, we refer to Cantor and Hamilton (2007).

(Figure 2.1 about here)

In the present study, we are dealing with the correlation between business sectors. Consequently, we must consider each sector separately in order to assess its default risk and to test its dependence with the remaining sectors. To do so, we build up quarterly rating transition matrices for each business sector. The descriptive statistics relative to the quarterly number of ratings per sector are displayed in Table 2.2.

(Table 2.2 about here)

As noticed for pooled data, Figure 2.2 shows an increase over time in the number of the ratings per quarter for each sector (except transportation sector).

(Figure 2.2 about here)

A second remark concerns the "Miscellaneous sector". It includes all business sectors defined by Moody's with an average ratings per quarter less than 100. We establish such a rule to avoid dealing with estimated transition probabilities characterized by large confidence intervals.

2.3.2 Macro economic variables

Several past studies have reported the linkage between default risk and the business cycle. In the present study, we will test if the evaluation of default risk via the mobility metrics allows us to assess this linkage. To do so, we will consider some macroeconomic variables already cited in the literature. The first data consists on the classification of historical periods of the U.S. economy. The National Bureau of Economic Research (NBER) lists 16 quarters of recession and 85 quarters of expansion from January 1980 to April 2005. This business cycle classification was used, for instance, by Bangia, Diebold, Kronimus, Schagen and Schuermann (2002). By distinguishing expansion periods from recession periods, they conclude a statistically significant variation of the estimated migration matrices across the two cycles.

The real GDP growth is the most common variable used by past studies to measure the link between default risk and the business cycle. For instance, Koopman and Lucas (2005) use the real GDP growth as a proxy of the economy growth. This data is available from the U.S. Bureau of Economic Analysis (www.bea.gov).

A third group of variables is provided by the U.S. Bureau of Labor Statistics (www.bls.org): we obtain time series of the unemployment rate and the Consumer Price Index (CPI) from the first quarter of 1980 to the first quarter of 2005. The CPI time series is used to assess the inflation rate. Both time series (unemployment and inflation rates) should be determinants of the strength of the economy: low unemployment and inflation rates are usually the targets of economic policy.

Another macroeconomic variable worth reporting is the real interest rate. A high real interest rate generally induces difficulties in servicing the debt. Therefore, we will investigate the link between our mobility metrics and the real interest rate. This variable is derived from the U.S. 3 months T-bill rate provided by the International Monetary Fund and the inflation rate cited above.

Finally, we will consider a macroeconomic index previously used by Figlewski, Frydman and Liang (2006). These authors test whether the Chicago Fed National Activity Index (CFNAI hereafter) is a determinant of default risk. This index is provided monthly by the Chicago Federal Reserve and summarizes 85 existing monthly indicators of U.S. economic activity. A positive value of the CFNAI indicates a growth above the trend whereas a negative value corresponds to a growth below the trend.

2.4 Mobility measure selection

In order to deal with the dimensionality problem mentioned in the introduction, we have to summarize the quarterly observed migration matrix of each sector into one scalar. Thus, we will estimate, in a first step. the transition matrix for each sector and for each quarter. The cohort method will be used to estimate such a matrix[2]. The second step consists on calculating the mobility index for the estimated transition matrix. We will obtain, then, a time series of the mobility index for each U.S. business sector.

However, a large number of mobility measures are cited in the literature. Therefore, it would be judicious to select a unique and "informative" metric for our analysis. If we focus on the cell by cell distance metrics, we easily observe that the *DC1* and *DC2* measures, defined in (2.1) and (2.2) respectively, should be discarded. These two metrics take into account all the elements of the mobility matrix defined in (2.4) without distinguishing between upgrades and downgrades. Thus, we can not conclude if the considered business sector is mainly upgrading or downgrading during a specific quarter: this is a crucial information when dealing with default risk.

Also, the *DEVE* measure, defined in (2.9), can not be useful for our study. This measure permits only the comparison between two matrices and does not provide information about the direction of migrations. Thus, we will keep, as a first selection, the DC3 metric, the eigenvalue based metrics (namely, *DEVA1*, *DEVA2* and *DEVA3*) and the singular value based metric (*DSV*) defined by Jafry and Schuermann (2004).

It is worth noting that the three eigenvalue based metrics should lead to the same conclusions since they are based on the second largest eigenvalue of the transition matrix. Therefore, we will keep only one of these metrics for analysis in the following selection. Retaining the *DEVA3* metric from this category looks the most judicious.

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[2] By excluding the withdrawn issues, as mentioned above.

Our choice is just explained by the easy interpretation of the scalar obtained via such a metric. For our case, a *DEVA3* standing at 20 indicates that we need 5 years (20 quarters) to observe 50% of the initial firms of the sector defaulting (if we assume a Markovian behavior of the rating migration).

Finally, we have to select one of the remaining metrics (*DC3*, *DEVA3* and *DSV*). To do so, we will test the performance of these metrics via three transitions matrices, namely *P1*, *P2* and *P3* defined as follows:

$$P1 = \begin{bmatrix} 0.85 & 0.09 & 0.04 & 0.02 \\ 0.02 & 0.94 & 0.03 & 0.01 \\ 0.03 & 0.06 & 0.85 & 0.06 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{P}2 = \begin{bmatrix} 0.85 & 0.09 & 0.04 & 0.02 \\ 0.02 & 0.94 & 0.03 & 0.01 \\ 0.03 & \mathbf{0.09} & \mathbf{0.82} & 0.06 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{P}3 = \begin{bmatrix} 0.85 & 0.09 & 0.04 & 0.02 \\ 0.02 & \mathbf{0.90} & 0.03 & \mathbf{0.05} \\ 0.03 & 0.06 & 0.85 & 0.06 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where the values in bold indicate the modified transition probabilities with reference to those observed in *P1*. Thus, we notice that *P2* (*respec. P3*) reveals more upgrades (*respec*. downgrades), if compared to *P1*.

Next, we evaluate the mobility of each transition matrix in three different ways. In Table 2.3, we display the calculations of the mobility index of *P1*, *P2* and *P3* via the three remaining techniques.

(Table 2.3 about here)

If we consider *P1* as our reference, we remark that the *DSV* metric increases when applied to *P2*. This same metric increases also when applied to *P3*. In other words, the *DSV* metric does not allow us to distinguish between a matrix characterized by more upgraded firms (*P2* compared to *P1*) and a matrix containing more downgraded firms (*P3* compared to *P1*). Thus, the *DSV* metric should be discarded since it looks inappropriate to assess the default risk.

However, *DC3* and *DEVA3* perform well when applied to different transition matrices. More upgrades should imply a longer time for firms to default and a higher *DEVA3* metric. The results obtained in Table 2.3 indicate a higher (*respec.* lower) *DEVA3* if we compare the transition matrices *P1* and P2 (*respec. P1* and *P3*).The analysis of the results obtained through the *DC3* metric allows for the same conclusions. In fact, this last metric, by its construction, should distinguish between more upgrades and more downgrades. An increase in upgrades will imply a higher *DC3* measure whereas an increase of downgrades will imply a lower *DC3*. Moreover, this mobility index treats the observed migrations differently: an issue migrating from Aaa to B will have an impact on the *DC3* metric five times greater than an issue migrating from Aaa to Aa. Table 2.3 demonstrates the increase in *DC3* when we compare *P2* to *P1*. The opposite result is obtained when considering *P3* and *P1*. Thus, we conclude that *DC3* is a useful metric to assess the default risk of a whole group, namely the firms belonging to one of the U.S. business sectors.

Theoretically, both *DEVA3* and *DC3* would be appropriate metrics. However, when confronted to our data, the *DEVA3* presents some important shortcomings. Here is a sample drawn from our database that will help us explaining our selection of the *DC3* as the "best" metric and reasons compelling us to discard *DEVA3*.

The estimated transition matrix for the first quarter of 1989 and relative to the banking sector is the following:

	Aaa	Aa	A	Baa	Ba	B	Caa-C	D
Aaa	1.00	-	-	-	-	-	-	-
Aa	-	1.00	-	-	-	-	-	-
A	-	-	0.99	0.01	-	-	-	-
Baa	-	-	0.04	0.92	0.04	-	-	-
Ba	-	-	-	0.10	0.86	0.04	-	-
B	-	-	-	-	-	0.85	0.15	-
Caa-C	-	-	-	-	-	-	0.50	0.50
D	-	-	-	-	-	-	-	1.00

The eigenvalues of this matrix are:

0.82	0.95	0.99	**1.00**	**1.00**	0.85	0.50	**1.00**

We notice three absorbent states (Aaa, Aa and default) in this specific quarter and, consequently, three eigenvalues equal to unity. This statement implies a misleading interpretation of the *DEVA3* metric. Recall that this index was built to gauge the convergence rate to the absorbent state. Then, this *DEVA3* metric will give an idea about the convergence rate to all absorbent states. For our case, it will be impossible to assess the evolution of the default risk through this mobility metric.

Having selected the mobility metric for our study, we apply it to the estimated transitions matrices for each U.S. business sector. The descriptive statistics of the obtained time series are displayed in Table 2.4.

(Table 2.4 about here)

We notice a negative mean of the selected mobility metric for each business sector. We should not be surprised by such a result. Indeed, a negative *DC3* indicates a larger proportion of firms migrating to lower ratings compared to the proportion of upgraded firms: a consequence of the unique absorbent state which corresponds to default. After a long horizon, all existing firms will default, even those rated Aaa presently[3]. Also, we reject the normality hypothesis for all times series (each business sector separately or pooled data) at the 1% significance level. Such result is provided by Lilliefors test which looks more suitable for small sample data.

Figure 2.3 below displays the evolution of the *DC3* metric during the 101 quarters considered for each U.S. business sector.

(Figure 2.3 about here)

These figures show a spike for the energy sector during the second quarter of 1988: the *DC3* metric stands at 1.889. This is mainly due to 2 firms migrating from the Caa-C class to the Baa class within the period[4]. A second remark concerns the lowest *DC3* metric for the banking sector. Our data reveals 8 banks defaulting during the first quarter of 1990. Moreover, all banks rated Caa-C at the beginning of the quarter fall into default at the end of the quarter: that was a consequence of the junk bonds crisis observed during 1989-1990.

Finally, since we are analyzing interactions among the different U.S. business sectors, it is judicious to evaluate the dependence between mobility metrics relative to these sectors. Table 2.5 exhibits Pearson's linear correlation coefficients (upper elements) and Kendall's taus (lower elements).

(Table 2.5 about here)

Based on Table 2.5, we can conclude that the highest dependence is observed between the consumer products sector and industrial sector. Also, we notice that the

[3] From our sample, we notice 49 issuers rated Aaa during January 1980 and not withdrawn during the 25 following years. Only 4 of these issuers keep the same rating in January 2005. The remaining issuers defaulted or downgraded.

[4] There is a total of 3 firms belonging to the energy sector and rated Caa-C at the beginning of the considered quarter.

majority of sectors show a positive dependence: only few sectors are negatively correlated and a coefficient nearing zero is generally obtained for these negative correlations. Thus, we should take an interest to analyze the possible default contagion between business sectors.

2.5 Univariate analysis of mobility indexes

In the present section, we will test if the selected mobility metric, namely the *DC3* metric, captures a crisis transmission phenomenon within the same business sector. Then, we discuss possible links between the business cycle and the default cycle.

2.5.1 Crisis transmission

We suspect that more downgrades for issues belonging to a business sector should induce more downgrades in the future for the same sector. In other words, the mobility metric of *t*-th quarter should have an impact on future mobility metrics. Such feature could be explained by rating momentum[5] documented by Carty and Fons (1994) and Duffie and Singleton (2003). A second possible explanation relates to crisis transmission phenomenon within the same business sector: a downgrade of an issue may induce downgrades for others issues belonging to the same business sector, as a result of business linkage.

In order to verify the autocorrelation of observed *DC3* metrics for each business sector, we estimate the following model:

$$DC3_t = \beta_0 + \beta_1 DC3_{t-1} + \varepsilon_t \qquad \text{(R2.1)}$$

where $DC3_t$ corresponds to the mobility metric for a specific business sector during the t-th quarter.

The results of the regression (R2.1) are displayed in Table 2.6:

[5] Consecutive downgrades (or upgrades) for the same issuer.

(Table 2.6 about here)

The results shown in Table 2.6 indicate that 5 of the 9 business sectors display a statistically significant positive autocorrelation. Also, none of the remaining sectors display a significantly negative dependence between current and previous rating migration.

Results in Table 2.6 do not distinguish between both possible effects (rating momentum and crisis transmission). However, the crisis transmission effect should be much higher for our case. It is rare to observe a rating change for a specific issue during two consecutive quarters. By investigating our database, we found 7799 rating changes during the 101 quarters. Only 285 issuers of them changed their ratings during two consecutive periods.

A second effect worth analyzing concerns asymmetric effects of a rating change. We suspect that effects of downgrades are more pronounced than effects of upgrades: several empirical studies show higher correlations during turbulent periods (Das et al.(2006)). Thus, we propose to test if the selected mobility metric captures this asymmetry via a two step methodology. As a first step, we estimate the models (R2.2.1) and (R2.2.2) for two different cases: the specific business sector was mainly downgraded during the previous quarter ($DC3_{t-1}$<0) or mainly upgraded ($DC3_{t-1}$>0).

$$DC3_t = \beta_{0,1} + \beta_{1,1} DC3_{t-1} + \varepsilon_{1t} \quad \text{if } DC3_{t-1} < 0 \qquad \text{(R2.2.1)}$$

$$DC3_t = \beta_{0,2} + \beta_{1,2} DC3_{t-1} + \varepsilon_{2t} \quad \text{if } DC3_{t-1} > 0 \qquad \text{(R2.2.2)}$$

The results of both regressions are displayed in the Table 2.7.

(Table 2.7 about here)

The second step for testing rating momentum asymmetry consists to check the following hypothesis:

$$H_0\text{: } \beta_{1,1} = \beta_{1,2}$$

$$H_1\text{: } \beta_{1,1} \neq \beta_{1,2}$$

Chow (1960) demonstrates that testing the equality among two coefficients in two regressions needs the examination of the statistic F defined in (2.10).

$$F = \frac{Q_1 - Q_2}{\dfrac{Q_2}{m+n-2p}} \qquad (2.10)$$

where Q_1 designates the sum of squares of the residuals (for both regressions R2.2.1 and R2.2.2) under H_0. Q_2 corresponds to the sum of squares of the residuals under H_1 for both regressions. m (*respec.* n) represents the number of observations of the regression R2.2.1 (*respec.* R2.2.2). Finally, p corresponds to the total number of explanatory variables of each regression.

Chow (1960) proves that, under H_0, the statistic F follows a Fisher-Snedecor distribution with 1 and ($m+n-2p$) degrees of freedom. The application of this Chow (1960) test to our sample leads to the following results:

(Table 2.8 about here)

The results show that the asymmetric effects of a rating change are statistically significant only if we consider all sectors pooled together or if we focus on the energy sector[6].

Finally, we remark that the 5 business sectors displaying a rating autocorrelation in Table 2.6 correspond to the sectors with a statistically significant downgrade rating

[6] In order to check if the asymmetric effect for the "all sectors" is only due to the energy sector, we apply the Chow (1960) test to all sectors pooled together except the energy sector. The results show up the existence of this asymmetric effect even after excluding the energy sector.

autocorrelation in Table 2.7. Also, none of them shows a statistically significant upgrade rating autocorrelation (except the banking sector at 10% level).

2.5.2 Mobility metrics and the business cycle

The linkage between rating transitions and the business cycle is well documented by previous studies. Nickell et al.(2000), among others, conclude that the transition probabilities differ from peak to trough periods. We propose the following regression to test if these linkages are still captured when we sum up the whole transition matrix by way of the selected mobility index:

$$DC3_t = \alpha_0 + \alpha_1 RGDPC_t + \varepsilon_t \qquad \text{(R2.3)}$$

where *RGDPG* corresponds to the real GDP growth. Table 2.9 summarizes the results of regression (R2.3):

(Table 2.9 about here)

We notice that the selected mobility metric conforms to the previous studies: an increase of the real GDP growth implies, usually, a lower default risk. The rating migration of four U.S. business sectors (from the 9 studied) is positively linked to the business cycle with a 95% confidence level. None of the remaining business sectors shows a statistically negative relation between the mobility index and the real GDP growth.

Adding three explanatory variables to the regression (R2.3) does not alter our previous conclusions. Indeed, if we take into account the real interest rate (*Rinter*), the unemployment rate (*Uempl*) and the inflation rate (*Infl*) to test the linkage between the business cycle and default risk, we can estimate the following model:

$$DC3_t = \alpha_0 + \alpha_1 Rinter_t + \alpha_2 Uempl_t + \alpha_3 RGDPC_t + \alpha_4 Infl_t + \varepsilon_t \qquad \text{(R2.4)}$$

The (R2.4) estimation results are shown in Table 2.10.

(Table 2.10 about here)

We conclude that the statistically positive link between *DC3* and the real GDP growth is obtained for the same four U.S. business sectors. Again, none of the remaining business sectors becomes more risky in presence of a positive real GDP growth. An increase of the real interest rate will imply, notably, a decrease of the mobility index relative to the banking sector and thus an increase of its default risk. Finally, we remark the negative effects of inflation on the solvability of the banking sector. The same effect is not observable for the remaining sectors at the 95% confidence level.

In order to confirm our conclusions about the dependence between the business cycle and the default cycle, we propose another proxy for the state of the economy, namely the Chicago Fed National Activity Index (CFNAI).

$$DC3_t = \alpha_0 + \alpha_1 CFNAI_t + \varepsilon_t \qquad \text{(R2.5)}$$

The results of regression (R2.5) are the following:

(Table 2.11 about here)

Again, the *DC3* is positively related to the CFNAI for four business sectors. If compared to the (R2.3) results, the only difference concerns the banking and the miscellaneous sectors. The first becomes significantly dependent to the new proxy whereas the second becomes independent. Such similarity is mainly explained by the correlation among the two proxies (the real GDP growth and the CFNAI) standing at 78.5%.

Finally, we propose to follow Bangia et al.(2002) methodology to assess the evolution of our mobility metric through the contraction and expansion cycles defined by the National Bureau of Economic Research (NBER). The mean and the standard deviation of the *DC3* metric are first estimated for both cycles. Then, we test if the mean differs statistically among the cycles through the following *t-stat*:

$$t - stat = \frac{M_c - M_e}{\sqrt{\frac{s_c^2}{N_c} + \frac{s_e^2}{N_e}}} \qquad (2.11)$$

where M_i, s_i^2 and N_i correspond, respectively, to the sample mean, the sample variance and the sample size of the cycle $i(i = c, e)$. Table 2.12 displays the results.

(Table 2.12 about here)

Only two business sectors show a statistically different mean if we compare expansion and contraction periods. However, none of the remaining sectors displays a contradiction to the results of Bangia et al.(2002). In other words, we never observe more upgrades (higher *DC3* indices) during the contraction periods.

2.6 Migration dependence

Having proved the utility of the *DC3* as a measure of default risk for the U.S. business sectors, we will focus on the dependence that might exist between these entities. We suspect the existence of two patterns of linkage: a cross-sectional dependence explained by common factors affecting all the sectors at the same time (such as the economic policy, the expansion and contraction periods...) and a credit contagion phenomena, that is to say, a transmission of the shock observed in a particular business sector to one (or more) of the remaining sectors. This second feature of dependence is due to the business links between firms belonging to different sectors.

At first sight, it seems judicious to use a vector autoregressions (VAR) model to estimate these two dependences. Such a model is generally used to capture comovements among many variables. However, we will favor a more generalized model that allows for regime-switching. This generalization is needed as a consequence of our previous results and those obtained by past studies: the default risk increases during economic downturns. Thus, we propose to expose, as a first step, some regime-switching VAR models that are applicable to our context and discuss their estimations. The empirical results will follow.

2.6.1 Regime-switching VAR models

The common feature of regime-switching VAR models consists on a K dimensional time series vector $y_t = (y_{1t}, \ldots, y_{Kt})$ defined conditional upon the regime $s_t \in \{1, \ldots, M\}$. The general form of such processes is the following:

$$y_t = \nu(s_t) + A_1(s_t) y_{t-1} + \cdots + A_p(s_t) y_{t-p} + B_1(s_t) x_t + \ldots + B_q(s_t) x_{t-q+1} + \varepsilon_t \quad (2.12)$$

where $\nu(s_t)$ is the vector of intercepts at regime s_t, $A_i(s_t)$ the matrix containing the autoregressive parameters at the same regime, ε_t an error term vector such that $\varepsilon_t | s_t \to IID(0, \sum(s_t))$, $x_t = (x_{1t}, \ldots, x_{Nt})$ the vector containing the N exogenous variables, and $B_j(s_t)$ the matrix containing the non autoregressive parameters at regime s_t.

The regime variable s_t can be observable or unobservable: an example of observable regimes is relative to *structural change models*. In such processes, the regime is determined as following:

$$s_t = \begin{cases} 1 & \text{if } t \le \tau_1 \\ 2 & \text{if } \tau_1 < t \le \tau_2 \\ \vdots & \\ M & \text{if } t > \tau_{M\text{-}1} \end{cases} \quad (2.13)$$

A second example of observable regimes concerns the *threshold autoregressive models*. In this case, the regime shifts are triggered by one or more exogenous variables crossing a specified threshold (c_i). In other words,

$$s_t = \begin{cases} 1 & \text{if } g(x_t) \leq c_1 \\ 2 & \text{if } c_1 < g(x_t) \leq c_2 \\ \vdots & \\ M & \text{if } g(x_t) > c_{M\text{-}1} \end{cases} \tag{2.14}$$

Such process could be used in our context: the rating migration is linked to some macroeconomic variables such as the real GDP growth, NBER cycles, and the CFNAI. However, Amato and Furfine (2004) found "*no evidence that credit ratings are unduly influenced by the business cycle*". Thus, we will favor a process with an unobservable regime changes called Markov-Switching VAR model (MSVAR hereafter). In these models, first introduced by Krolzig (1997), the unobservable regime variable $s_t \in \{1,\ldots,M\}$ is generated by an ergodic Markov chain defined by the transition probabilities p_{ij} with:

$$\begin{aligned} p_{ij} &= \Pr\left(s_{t+1} = j \middle| s_t = i\right) \\ \sum_{j=1}^{M} p_{ij} &= 1 \quad , p_{ij} \geq 0 \quad \forall\, i, j = \{1,\ldots,M\} \end{aligned} \tag{2.15}$$

A major advantage of MSVAR models consists on their flexibility since we can allow for some non switching estimators. For example, we can estimate an MSVAR model by assuming invariant autoregressive parameters and invariant intercepts of equation (2.12). Such particular specification is known as MSH VAR model. Also, we can consider the least constrained specification: the one allowing for varying intercepts, autoregressive parameters and conditional variance, usually known as the MSIAH

VAR specification[7]. The notation used to specify the various Markov Switching models follows this rule:

* *I* denotes a varying intercept term
* *A* denotes a varying autoregressive parameters
* *H* denotes heteroskedasticity (varying variances and covariances)

The Akaike Information or Schwarz criteria are generally used to select one of the available specifications and/or the number of lags to be considered. However, the optimal number of regimes (*M*) is obtained according to Davies (1987) procedure[8].

2.6.2 Model estimation

Once we select the MSVAR specification, the number of lags and the number of regimes, we can estimate the obtained model by maximizing its likelihood function. Such maximization entails the implementation of the Expectation-Maximization (EM hereafter) algorithm introduced by Dempster, Laird and Rubin (1977) as a consequence of the unobservable regime variable s_t. Each iteration of this EM algorithm is composed into two steps. The *expectation* step consists on computing the expected likelihood by replacing the unknown parameters by their estimated value obtained during the previous iteration. At the *maximization* step, an estimate of the parameters is derived by maximizing the expected likelihood function found during the *expectation* step. Then, these estimated parameters will be used at the beginning of the next iteration and so on.

At the end of iterations, we can derive the filtered regime probabilities which refer to inferences about the regime variable s_t conditional on information up to time *t*, the smoothed regime probabilities reporting the inference about s_t by using all the information of the sample and the one-step predicted regime probabilities exposing the inference about s_t conditional on information up to *t-1*.

[7] See Krolzig (1997) for more details on MSVAR specifications.
[8] Garcia and Perron (1996) describe concisely this procedure.

2.6.3 Empirical results

We will start by focusing on the U.S. banking and industrial sectors in this empirical part. First, we will estimate an MSIAH VAR specification with one lag, two regimes and no exogenous variables (MSIAH(2) VAR(1)). We set a minimum percentage change of 10^{-6} of the likelihood function as condition to let the EM algorithm iterate again. Finally, we assume Gaussian error terms to derive the likelihood function. In other words, the model exhibited in (2.12) will be simplified to:

$$\begin{cases} y_{b,t} = v_{b,s} + a_{bb,s} y_{b,t-1} + a_{ib,s} y_{i,t-1} + \varepsilon_{b,t} \\ y_{i,t} = v_{i,s} + a_{bi,s} y_{b,t-1} + a_{ii,s} y_{i,t-1} + \varepsilon_{i,t} \end{cases} \tag{2.16}$$

with *s*={1,2} indicating the regime variable, $y_{b,t}$ and $y_{i,t}$ designate the observable *DC3* metric for the banking and industrial sectors at the *t*-th quarter. The vector of error terms $\left(\varepsilon_{b,t};\varepsilon_{i,t}\right)'$ follows, conditionally, a multivariate centered Gaussian distribution:

$$\begin{gathered} \begin{bmatrix} \varepsilon_{b,t} | s \\ \varepsilon_{i,t} | s \end{bmatrix} \rightarrow NID\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \sum(s) \right) \\ \sum(s) = \begin{bmatrix} \sigma_{b,s}^2 & \rho_{bi,s}\sigma_{b,s}\sigma_{i,s} \\ \rho_{bi,s}\sigma_{b,s}\sigma_{i,s} & \sigma_{i,s}^2 \end{bmatrix} \end{gathered} \tag{2.17}$$

$\sum(s)$ denotes the variance covariance matrix of the error terms during a specified regime. The estimation results are displayed in Table 2.13 following:

(Table 2.13 about here)

At a first sight, we can notice a regime of low correlation (*s*=1) among both considered sectors and a regime of high correlation. The statistical significance of the autoregressive parameters will be discussed later in order to assess the contagion effects. The Davies (1987) test shows a superiority of the MSIAH(2) VAR(1) model compared to the linear VAR model (no switching regime). Such superiority is observed also via the likelihood ratio linearity test. This statistic equals:

$$LR = 2\ln\left(\frac{\text{Likelihood function of regime switching model}}{\text{Likelihood function of linear model}}\right) \qquad (2.18)$$

Two additional informations are obtained during the previous estimation: the transition probabilities defined in (2.15) and the inferences about the regime variable. For the MSIAH(2) VAR(1) specification, the migration from one regime to the other is determined by the transition matrix P:

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = \begin{bmatrix} 0.57 & 0.43 \\ 0.37 & 0.63 \end{bmatrix}$$

The second set of information consists on the filtered, smoothed and predicted regime probabilities. We display some of these inferences in Figure 2.4:

(Figure 2.4 about here)

Given these probabilities, we notice the difference between the default cycle and the business cycle: some contraction quarters (defined by the NBER) belong to the low correlation regime such as the period from the third quarter of 1990 to the first quarter of 1991. However, the recession of the first three quarters of 2001 belong to the high correlation regime. Thus, our results confirm the conclusions of Amato et al.(2004): the business cycle may differ from the default cycle.

A last point worth discussing relates to the significance of the autoregressive parameters or more generally the impact of a shock observed in a particular business sector to all sectors during the next periods. Such analysis is usually done by the

Impulse Response Functions (IRF hereafter) analysis for the VAR models. However, testing the significance of the autoregressive parameters or the IRF needs the use of simulation techniques: the non normality[9] of error terms prevents us using the standard *t*-statistics to test the parameters significance.

Recall that the IRF conditional on regime *s* and based on the model (2.16) are the following:

$$\begin{bmatrix} IRF_{bb,t,s} & IRF_{ib,t,s} \\ IRF_{bi,t,s} & IRF_{ii,t,s} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} * \begin{bmatrix} a_{bb,s} & a_{ib,s} \\ a_{bi,s} & a_{ii,s} \end{bmatrix}^{t} \qquad (2.19)$$

where $IRF_{ib,t,s}$ corresponds to the effect after t quarters of a unitary shock in the industrial sector on the banking sector, given the regime s. By analogy, we define the remaining terms of (2.19).

Then, by applying the simulation methodology developed by Ehrmann, Ellison and Valla (2003), where we consider 500 simulations and 68% confidence intervals, we obtain the IRF for the next 9 quarters. As shown in Figure 2.5 below, we notice that a positive shock in the banking sector will imply a significant increase of the *DC3* metric of the same sector during the next 6 quarters conditional on being in the low correlation regime. However, the effects of the same shock will vanish quickly if we consider the high correlation regime. Moreover, we notice the presence of significant contagion effects during the high correlation regime: an increase of default risk in the industrial sector will induce a significant increase of the default risk relative to the banking sector. Simultaneously, an increase of default risk in the banking sector will lead to a decrease of the default risk relative to industrial sector. Such simultaneity prevents a snowball effect between banking and industrial sectors.

[9] In our case, we have a mixture of normals since we assume that each error term follows a Gaussian distribution, conditional on regime *s*.

Finally, our results confirm the univariate analysis of section 2.5: a positive shock in one sector never implies a decrease of the *DC3* metric relative to the same sector for any regime and any next period.

(Figure 2.5 about here)

A key assumption used in this empirical part concerns the use of a particular specification of MSVAR models, namely the MSIAH(2) VAR(1). In others words, we consider a MSVAR model with one lag, two regimes and intercepts, autoregressive parameters and standard errors varying with the endogenously defined regimes. We propose to test, by use of Schwarz criterion, if this selected specification is the best to represent the observed mobility metrics of the U.S. banking and industrial sectors. Table 2.14 following sums up the results relative to alternative specifications:

(Table 2.14 about here)

Thus, we notice the superiority of the MSIAH(2) VAR(1) specification if compared to alternative MSVAR models since it shows the minimal Schwarz criterion. Adding exogenous variables (such as the real GDP change) does not improve our results[10].

The same analysis held above could be applied to other couples of sectors. For instance, the estimation of various MSVAR specifications for the U.S. energy and industrial sectors supports the superiority of the MSIH(2) VAR(1) model. As shown in Table 2.15, the minimal Schwarz criterion is obtained by assuming constant autoregressive parameters, varying intercept terms, heteroskedasticity and a lag equal to one.

(Table 2.15 about here)

[10] For brevity, we did not include the complete results in our study. They could be provided upon request.

Recall that the MSIH(2) VAR(1) specification applied to the U.S. energy and industrial sectors is expressed as following:

$$\begin{cases} y_{e,t} = \nu_{e,s} + a_{ee} y_{e,t-1} + a_{ie} y_{i,t-1} + \varepsilon_{e,t} \\ y_{i,t} = \nu_{i,s} + a_{ei} y_{e,t-1} + a_{ii} y_{i,t-1} + \varepsilon_{i,t} \end{cases} \tag{2.20}$$

where s={1,2} indicates the regime variable, $y_{e,t}$ and $y_{i,t}$ designate the observable *DC3* metric for the U.S. energy and industrial sectors at the *t*-th quarter. The vector of error terms $(\varepsilon_{e,t};\varepsilon_{i,t})'$ is defined by analogy to (2.17). The estimation results of (2.20) are following:

(Table 2.16 about here)

The IRF analysis for both U.S. business sectors confirms that an increase of default risk in the industrial sector will induce a significant decrease of the default risk relative to the energy sector during the following year. Moreover, the effects of a positive shock in the energy sector will statistically vanish after one semester. Figure 2.6 below displays in details this IRF analysis.

(Figure 2.6 about here)

A final application of our methodology is applied to banking and energy sectors. The Schwarz criterion applied to these two sectors indicates a superiority of an MSH(2) VAR(1) specification: constant autoregressive parameters, constant intercept terms, heteroskedasticity and a lag equal to one.

(Table 2.17 about here)

In other words, the model to estimate to assess migration dependence between banking and energy sectors is following:

$$\begin{cases} y_{b,t} = v_b + a_{bb} y_{b,t-1} + a_{eb} y_{e,t-1} + \varepsilon_{b,t} \\ y_{e,t} = v_e + a_{be} y_{b,t-1} + a_{ee} y_{e,t-1} + \varepsilon_{e,t} \end{cases} \tag{2.21}$$

with $y_{b,t}$ and $y_{e,t}$ designate the observable *DC3* metric for the U.S. banking and energy sectors at the *t*-th quarter. The vector of error terms $(\varepsilon_{b,t};\varepsilon_{e,t})'$ is defined by analogy to (2.17). The estimation results of (2.21) are the following:

(Table 2.18 about here)

From IRF analysis displayed in Figure 2.7, we notice that a positive shock in either business or energy sectors does not significantly affect the other sector. However, a statistically significant effect is observed in the same business sector: a shock in the banking sector will vanish after three quarters while a shock in the energy sector will vanish after one year.

(Figure 2.7 about here)

2.7 Conclusions

In this paper, we proposed a technique allowing to measure simultaneously the migration correlation and the default contagion among several groups of issuers. Furthermore, this technique avoids the dimensionality problem since it sums up the whole transition matrix to a scalar, namely the mobility metric. As a first step, we a held an univariate analysis of these mobility metrics. The results support crisis transmission within the same business sector. We provide evidence of linkage between the business and default cycles for several U.S. business sectors during the period 1980-2005. Then, we estimate the dependence among some of these business sectors by use of a regime switching VAR model. For instance, we identified two regimes when focusing on the U.S. banking and industrial sectors: a regime of low correlation (7%) between both sectors and a second regime of higher correlation (15.58%). Moreover, we noticed a difference between the business cycles defined by the NBER and the default cycles identified endogenously by the MSIAH(2) VAR(1) model. Finally, the analysis by impulse response functions allows us to assess the persistence of a shock in a particular sector to the two considered sectors. For example, we observed that an increase of the default risk in the industrial sector will imply a significant increase of the default risk in the banking sector during the next quarter, conditional on being at the high correlation regime. Thus, a lot of attention should be accorded by regulators and investors to the financial institutions if they remark more frequent downgrades of the industrial issuers.

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Table 2.1: Distribution of withdrawals, defaults and new issues

This table presents the distribution of withdrawals, defaults and new issues with reference to credit rating classes

	Aaa	Aa	A	Baa	Ba	B	Caa-C	Total
Withdrawals	54	261	550	561	851	834	229	**3340**
Defaults	0	0	2	8	86	454	523	**1073**
New issues	108	481	921	902	1433	1913	341	**6099**

Table 2.2: Distribution of quarterly number of ratings per business sector

Table 2.2 reports descriptive statistics of the total number of ratings for each business sector and each quarter.

Business sector	**Average**	**Stand. dev**	**Maximum**	**Minimum**
Banking	179.14	79.88	285	48
Industrial	434.05	132.75	686	246
Financial non bank	294.89	133.61	629	65
Energy	122.64	37.21	189	72
Consumer products	136.65	37.40	209	78
Technology	182.94	71.68	336	84
Transportation	100.31	11.80	123	77
Utilities	323.55	28.65	394	293
Miscellaneous*	288.80	98.26	437	125
All sectors	**2062.98**	**590.69**	**3017**	**1125**

**The Miscellaneous category includes Hotel gaming and leisure, Retail and Media sectors.*

Table 2.3: Comparison of the mobility metrics

Table 2.3 sums up the mobility metrics obtained from matrices *P1*, *P2* and *P3* by applying one of three methods analyzed, namely *DSV*, *DEVA3* and *DC3*.

	P1	*P2*	*P3*
DSV	0.1005	0.1089	0.1144
DEVA3	32.036	34.835	14.420
DC3	-0.200	-0.170	-0.280

Table 2.4: Descriptive statistics of the selected mobility measure

Table 2.4 provides descriptive statistics of the *DC3* mobility metric for each business sector. The last column indicates *p-value* of Lilliefors test. This test rejects the null hypothesis of normality at the 1% significance level for values lower than 0.01. Our preference to the Lilliefors test is induced by the small size of our sample.

Business sector	**Mean**	**S. dev**	**Min**	**Max**	**Skew**	**Kurt**	***p-value***
Banking	-0.083	0.442	-1.890	0.863	-1.470	6.839	<0.001
Industrial	-0.173	0.233	-1.025	1.074	0.556	11.565	<0.001
Financial NB	-0.138	0.341	-1.323	1.024	-0.325	5.959	<0.001
Energy	-0.117	0.356	-1.194	1.889	1.047	12.282	<0.001
Consumer Prod.	-0.141	0.220	-0.933	0.871	-0.399	9.034	<0.001
Technology	-0.175	0.316	-1.234	0.878	-0.804	5.725	<0.001
Transportation	-0.146	0.368	-1.619	1.104	-0.952	7.839	<0.001
Utilities	-0.022	0.432	-1.546	1.525	-0.294	6.148	<0.001
Miscellaneous	-0.187	0.249	-1.365	0.525	-1.858	9.414	<0.001
All sectors	-0.172	0.214	-1.151	0.592	-0.845	8.117	0.002

Table 2.5: Mobility metrics dependence (%)

As a consequence of normality rejection, we provide two measures of dependence, namely Pearson's linear correlation coefficients (upper elements) and Kendall's taus (lower elements).

Business sector	**(1)**	**(2)**	**(3)**	**(4)**	**(5)**	**(6)**	**(7)**	**(8)**	**(9)**
Banking (1)	-	12	15	3	8	10	9	9	23
Industrial (2)	14	-	18	9	28	15	9	-3	22
Financial NB (3)	14	20	-	-11	-20	8	18	-1	2
Energy (4)	12	7	-4	-	10	8	-4	-4	-15
Consumer prod (5)	2	24	-5	11	-	17	4	-15	1
Technology (6)	5	17	7	7	14	-	2	9	8
Transportation (7)	0	10	14	-4	7	8	-	12	19
Utilities (8)	-2	2	7	-6	-5	17	6	-	-7
Miscellaneous (9)	8	18	6	-6	17	7	12	1	-

Table 2.6: Rating momentum & Crisis transmission

Table 2.6 reports the results of regression (R2.1) for each business sector. The aim is to test for possible autocorrelation of *DC3* metrics.

$$DC3_t = \beta_0 + \beta_1 DC3_{t-1} + \varepsilon_t \qquad \text{(R2.1)}$$

	$\hat{\beta}_0$ *(t-stat)*	$\hat{\beta}_1$ *(t-stat)*	R^2
Banking	-0.036	0.557***	31%
	(-0.96)	*(6.64)*	
Industrial	-0.126***	0.262***	7%
	(-4.43)	*(2.68)*	
Financial non bank	-0.138***	0.002	0.1%
	(-3.69)	*(0.02)*	
Energy	-0.074**	0.375***	14%
	(-2.10)	*(4.00)*	
Consumer products	-0.149***	-0.046	0.2%
	(-5.66)	*(-0.46)*	
Technology	-0.131***	0.257***	6%
	(-3.71)	*(2.63)*	
Transportation	-0.109***	0.264***	7%
	(-2.84)	*(2.72)*	
Utilities	-0.020	0.065	0.4%
	(-0.45)	*(0.63)*	
Miscellaneous	-0.195***	-0.035	0.1%
	(-6.20)	*(-0.35)*	
All sectors	-0.144***	0.162	3%
	(-5.26)	*(1.62)*	

*** Significant at the 1% level

** Significant at the 5% level

Table 2.7: Asymmetric autocorrelation

Table 2.7 provides the first step results to test asymmetric autocorrelation hypothesis Regressions (R2.2.1) and (R2.2.2) relate respectively to the downgrade case (*DC3*<0 during *t-1*) and upgrade case (*DC3*>0 during *t-1*)

$$DC3_t = \beta_{0,1} + \beta_{1,1} DC3_{t-1} + \varepsilon_{1t} \quad \text{if } DC3_{t-1} < 0 \qquad \text{(R2.2.1)}$$

$$DC3_t = \beta_{0,2} + \beta_{1,2} DC3_{t-1} + \varepsilon_{2t} \quad \text{if } DC3_{t-1} > 0 \qquad \text{(R2.2.2)}$$

	$DC3_{t-1} < 0$ *(downgrade)*			$DC3_{t-1} > 0$ *(upgrade)*		
	$\hat{\beta}_{0,1}$ *(t-stat)*	$\hat{\beta}_{1,1}$ *(t-stat)*	*N. obs*	$\hat{\beta}_{0,2}$ *(t-stat)*	$\hat{\beta}_{1,2}$ *(t-stat)*	*N. obs*
Banking	0.001	0.619***	48	0.018	0.358*	44
	(0.02)	*(4.42)*		*(0.27)*	*(1.66)*	
Industrial	-0.141***	0.244**	91	0.054	-0.101	9
	(-4.55)	*(2.21)*		*(0.33)*	*(-0.23)*	
Financial NB	-0.143**	-0.039	71	-0.195**	0.234	28
	(-2.41)	*(-0.25)*		*(-2.41)*	*(0.90)*	
Energy	0.021	0.649***	56	-0.019	0.081	38
	(0.24)	*(3.06)*		*(-0.73)*	*(1.07)*	
Cons. Prod	-0.169***	-0.090	76	-0.077	-0.242	15
	(-5.85)	*(-0.88)*		*(-1.07)*	*(-0.78)*	
Technology	-0.114**	0.298**	72	-0.139*	0.188	28
	(-2.33)	*(2.46)*		*(-1.77)*	*(0.52)*	
Transportation	-0.070	0.372***	65	-0.065	-0.050	23
	(-1.18)	*(2.83)*		*(-1.54)*	*(-0.41)*	
Utilities	-0.027	0.022	48	-0.045	0.156	52
	(-0.29)	*(0.11)*		*(-0.68)*	*(0.89)*	
Miscellaneous	-0.232***	-0.116	87	-0.025	-0.610*	11
	(-6.04)	*(-0.99)*		*(-0.43)*	*(-1.91)*	
All sectors	-0.123***	0.263**	88	-0.042	-0.684**	12
	(-3.57)	*(2.16)*		*(-0.68)*	*(-2.32)*	

*** Significant at the 1% level

** Significant at the 5% level

* Significant at the 10% level

Table 2.8: Chow (1960) test of asymmetric autocorrelation

Table 2.8 exhibits the second step results to test asymmetric correlation. A *p-value* larger than 0.01 is synonymous to the null hypothesis acceptance (no asymmetric correlation) at the 1% significance level. *F*-statistics are obtained via the sum of square residuals of regressions (R2.2.1) and (R2.2.2).

	Deg. freedom	**F-statistic**	***p-value***
Banking	(1 , 88)	0.828	0.3655
Industrial	(1 , 96)	1.746	0.1895
Financial non bank	(1 , 95)	0.788	0.3770
Energy	(1 , 90)	5.185	0.0252
Consumer products	(1 , 87)	0.351	0.5549
Technology	(1 , 96)	0.105	0.7462
Transportation	(1 , 84)	2.468	0.1199
Utilities	(1 , 96)	0.258	0.6126
Miscellaneous	(1 , 94)	0.793	0.3755
All sectors	(1 , 96)	6.676	0.0113

Table 2.9: Effects of the real GDP growth on the mobility index

In Table 2.9, we report results of regression (R2.3). The dependent variable is the mobility metric (*DC3*), and the explanatory variable corresponds to the real GDP growth (*RGDPG*).

$$DC3_t = \alpha_0 + \alpha_1 RGDPC_t + \varepsilon_t \qquad \text{(R2.3)}$$

	$\hat{\alpha}_0$ *(t-stat)*	$\hat{\alpha}_1$ *(t-stat)*	R²
Banking	-0.148**	2.172	2%
	(-2.38)	*(1.48)*	
Industrial	-0.250***	2.545***	11%
	(-7.95)	*(3.43)*	
Financial non bank	-0.208***	2.315**	4%
	(-4.35)	*(2.05)*	
Energy	-0.060	-1.905	3%
	(-1.19)	*(-1.61)*	
Consumer products	-0.141***	-0.015	<0.1%
	(-4.49)	*(-0.02)*	
Technology	-0.245***	2.301**	5%
	(-5.54)	*(2.21)*	
Transportation	-0.153***	0.223	<0.1%
	(-2.90)	*(0.18)*	
Utilities	-0.023	0.034	<0.1%
	(-0.37)	*(0.02)*	
Miscellaneous	-0.239***	1.717**	4%
	(-6.85)	*(2.09)*	
All sectors	-0.200***	0.943	2%
	(-6.60)	*(1.32)*	

*** Significant at the 1% level

** Significant at the 5% level

Table 2.10: Effects of some macro variables on the mobility index

Results of regression (R2.4) are displayed below. We consider four explanatory variables: the real interest rate (*Rinter*), the unemployment rate (*Uempl*), the real GDP growth (*RGDPG*) and the inflation rate (*Infl*).

$$DC3_t = \alpha_0 + \alpha_1 Rinter_t + \alpha_2 Uempl_t + \alpha_3 RGDPC_t + \alpha_4 Infl_t + \varepsilon_t \quad \text{(R2.4)}$$

	$\hat{\alpha}_0$ *(t-stat)*	$\hat{\alpha}_1$ *(t-stat)*	$\hat{\alpha}_2$ *(t-stat)*	$\hat{\alpha}_3$ *(t-stat)*	$\hat{\alpha}_4$ *(t-stat)*	R^2
Banking	-0.118	-5.863***	4.616	1.806	-4.882***	13%
	(-0.61)	*(-3.03)*	*(1.44)*	*(1.28)*	*(-2.85)*	
Industrial	-0.259**	-0.612	0.539	2.520***	-0.289	11%
	(-2.53)	*(-0.59)*	*(0.31)*	*(3.34)*	*(-0.32)*	
Financial NB	-0.183	-2.279	0.962	2.220**	-0.889	6%
	(-1.19)	*(-1.46)*	*(0.37)*	*(1.96)*	*(-0.65)*	
Energy	0.405***	0.354	-9.142***	-1.873*	2.615*	17%
	(2.68)	*(0.23)*	*(-3.60)*	*(-1.68)*	*(1.93)*	
Cons. prod.	-0.166	0.828	0.255	-0.004	-0.239	1%
	(-1.63)	*(0.81)*	*(0.15)*	*(-0.01)*	*(-0.26)*	
Technology	-0.323**	4.022***	-1.295	2.488**	1.801	13%
	(-2.35)	*(2.89)*	*(-0.56)*	*(2.45)*	*(1.47)*	
Transportation	-0.226	0.753	-0.066	0.339	1.569	1%
	(-1.33)	*(0.44)*	*(-0.02)*	*(0.27)*	*(1.03)*	
Utilities	-0.061	1.502	-1.012	0.159	1.742	1%
	(-0.30)	*(0.74)*	*(-0.30)*	*(0.11)*	*(0.97)*	
Miscellaneous	-0.353***	-1.915*	2.314	1.719**	0.292	9%
	(-3.17)	*(-1.71)*	*(1.24)*	*(2.10)*	*(0.29)*	
All sectors	-0.148	-1.148	-0.505	0.916	0.138	4%
	(-1.51)	*(-1.16)*	*(-0.31)*	*(1.27)*	*(0.16)*	

*** Significant at the 1% level

** Significant at the 5% level

* Significant at the 10% level

Table 2.11: Effects of the CFNAI on the mobility index

Table 2.11 shows the links between the mobility index (*DC3*) and Chicago Fed National Activity Index (CFNAI). A positive value of this index indicates a growth of the U.S. economy above the trend whereas a negative value corresponds to a growth below the trend.

$$DC3_t = \alpha_0 + \alpha_1 CFNAI_t + \varepsilon_t \qquad \text{(R2.5)}$$

	$\hat{\alpha}_0$ *(t-stat)*	$\hat{\alpha}_1$ *(t-stat)*	R^2
Banking	-0.071	0.098*	3%
	(-1.62)	*(1.72)*	
Industrial	-0.158***	0.129***	18%
	(-7.42)	*(4.67)*	
Financial non bank	-0.123***	0.126***	8%
	(-3.72)	*(2.94)*	
Energy	-0.124***	-0.053	1%
	(-3.45)	*(-1.15)*	
Consumer products	-0.140***	0.010	0.1%
	(-6.31)	*(0.35)*	
Technology	-0.166***	0.082**	4%
	(-5.28)	*(2.03)*	
Transportation	-0.143***	0.026	0.3%
	(-3.84)	*(0.53)*	
Utilities	-0.027	-0.045	0.6%
	(-0.63)	*(-0.80)*	
Miscellaneous	-0.183***	0.036	1%
	(-7.29)	*(1.10)*	
All sectors	-0.168***	0.031	1%
	(-7.81)	*(1.10)*	

*** Significant at the 1% level
** Significant at the 5% level
* Significant at the 10% level

Table 2.12: Descriptive statistics of the mobility metric according to NBER cycles

Table 2.12 provides a comparison between mobility metrics (*DC3*) during contraction and expansion periods, as defined by the National Bureau of Economic Research (NBER). M_c and s_c^2 (*respec.* M_e and s_e^2) correspond to the mean and variance of (*DC3*) during contraction (*respec.* expansion) periods.

	Contraction		Expansion		Two sample *t*-test	
	M_c	s_c^2	M_e	s_e^2	M_c - M_e	*t-stat*
Banking	-0.164	0.494	-0.067	0.432	-0.097	-0.731
Industrial	-0.366	0.189	-0.137	0.223	-0.229***	-4.322
Financial NB	-0.265	0.224	-0.114	0.355	-0.151**	-2.228
Energy	-0.048	0.236	-0.130	0.374	0.083	1.155
Cons. Prod	-0.181	0.248	-0.134	0.215	-0.047	-0.706
Technology	-0.256	0.334	-0.159	0.313	-0.097	-1.071
Transportation	-0.249	0.432	-0.126	0.354	-0.123	-1.076
Utilities	0.111	0.426	-0.047	0.431	0.158	1.365
Miscellaneous	-0.245	0.374	-0.176	0.219	-0.069	-0.715
All sectors	-0.225	0.182	-0.162	0.219	-0.063	-1.221

*** Significant at the 1% level

** Significant at the 5% level

Table 2.13: MSIAH(2) VAR(1) estimation results for the banking and industrial sectors

Table 2.13 reports results of regression (16). Davies (1987) test allows comparison between our regime switching model and a linear VAR model.

$$\begin{cases} y_{b,t} = \nu_{b,s} + a_{bb,s} y_{b,t-1} + a_{ib,s} y_{i,t-1} + \varepsilon_{b,t} \\ y_{i,t} = \nu_{i,s} + a_{bi,s} y_{b,t-1} + a_{ii,s} y_{i,t-1} + \varepsilon_{i,t} \end{cases} \tag{2.16}$$

where $y_{b,t}$ and $y_{i,t}$ designate, respectively, the (*DC3*) of banking and industrial sectors during *t*-th quarter. *s* indicates the unobservable regime variable: 2 regimes are considered for our case.

	Regime 1	Regime 2
$\nu_{b,s}$	-0.0797	0.0526
$\nu_{i,s}$	-0.1716	-0.0631
$a_{bb,s}$	0.7403	0.1344
$a_{ib,s}$	0.0930	0.1130
$a_{bi,s}$	0.0561	-0.0591
$a_{ii,s}$	0.2081	0.3833
$\sigma_{b,s}$	0.4756	0.1097
$\sigma_{i,s}$	0.3067	0.0847
$\rho_{bi,s}$	0.0007	0.1558

log-likelihood : 12.4954

Schwarz criterion : 0.6711

Likelihood ratio linearity test : 91.9521

Davies test p-value : 0.0000

Table 2.14: Schwarz criterion for the banking and industrial sectors

Schwarz criterion allows comparison among various MS VAR models. The best specification should display the minimal statistic.

	lag=1	*lag=2*	*lag=3*
MSIAH	0.6711	0.8736	0.8475
Linear	1.0841	1.1600	1.2501
MSI	1.1790	1.0343	1.4073
MSA	1.1109	1.2643	1.3477
MSH	0.7747	0.7418	0.8082
MSIA	1.1127	1.0906	1.4388
MSIH	0.6804	0.6980	0.8527
MSAH	0.7237	0.9899	1.2294

Table 2.15: Schwarz criterion for the energy and industrial sectors

Table 2.15 compares various specifications of MS VAR models. When analyzing energy and industrial sectors, the MSIH(2) VAR(1) specification looks the most suitable: constant autoregressive parameters, varying intercept terms, varying variances and covariances and lag equal to one.

	lag=1	*lag=2*	*lag=3*
MSIAH	0.7413	0.8648	0.7568
Linear	0.8294	0.8800	1.0443
MSI	0.9246	1.0370	1.2213
MSA	0.7559	0.8902	0.7451
MSH	0.7585	0.7434	0.7041
MSIA	0.6547	0.6918	0.5650
MSIH	0.5433	0.6550	0.7443
MSAH	0.6865	0.6260	0.7732

Table 2.16: MSIH(2) VAR(1) estimation results for the energy and industrial sectors

Following conclusions from Table 2.15, we estimate model in (2.20). Statistic significance of autoregressive terms is analyzed in Figure 2.6.

$$\begin{cases} y_{e,t} = \nu_{e,s} + a_{ee} y_{e,t-1} + a_{ie} y_{i,t-1} + \varepsilon_{e,t} \\ y_{i,t} = \nu_{i,s} + a_{ei} y_{e,t-1} + a_{ii} y_{i,t-1} + \varepsilon_{i,t} \end{cases} \tag{2.20}$$

with $y_{e,t}$ and $y_{i,t}$ designate, respectively, the (*DC3*) of energy and industrial sectors during *t*-th quarter and *s* relates to the unobservable regime variable.

	Regime 1	Regime 2
$\nu_{e,s}$	-0.1630	-0.0729
$\nu_{i,s}$	-0.2277	-0.0611
a_{ee}	0.3610	
a_{ie}	-0.2346	
a_{ei}	0.0347	
a_{ii}	0.1469	
$\sigma_{e,s}$	0.2335	0.3866
$\sigma_{i,s}$	0.2937	0.0693
$\rho_{ei,s}$	0.1837	0.0373

log-likelihood : 4.6766

Schwarz criterion : 0.5433

Likelihood ratio linearity test : 50.8418

Davies test p-value : 0.0000

Table 2.17: Schwarz criterion for the banking and energy sectors

Table 2.17 indicates superiority of the MSH(2) VAR(1) specification: only variances and covariances vary among both regimes.

	lag=1	*lag=2*	*lag=3*
MSIAH	2.0205	1.8045	1.8914
Linear	1.7919	1.8997	1.9626
MSI	1.8211	1.9704	2.0767
MSA	1.8703	2.0963	2.0069
MSH	1.4024	1.4263	1.5763
MSIA	1.9926	1.9082	2.0604
MSIH	1.8517	1.5233	1.6131
MSAH	1.4595	1.5992	1.7821

Table 2.18: MSH(2) VAR(1) estimation results for the banking and energy sectors

Table 2.18 reports results of model in (2.21). IRF analysis in Figure (2.7) complements theses results by analyzing statistical significance of autoregressive parameters.

$$\begin{cases} y_{b,t} = \nu_b + a_{bb} y_{b,t-1} + a_{eb} y_{e,t-1} + \varepsilon_{b,t} \\ y_{e,t} = \nu_e + a_{be} y_{b,t-1} + a_{ee} y_{e,t-1} + \varepsilon_{e,t} \end{cases} \quad (2.21)$$

with $y_{b,t}$ and $y_{e,t}$ designate, respectively, the (*DC3*) of banking and energy sectors during *t*-th quarter and *s* relates to the unobservable regime variable.

	Regime 1	Regime 2
ν_b	0.0131	
ν_e	-0.0757	
a_{bb}	0.2592	
a_{be}	-0.0296	
a_{eb}	0.0156	
a_{ee}	0.4777	
$\sigma_{b,s}$	0.1250	0.6421
$\sigma_{e,s}$	0.2256	0.4641
$\rho_{eb,s}$	0.3110	0.1310

log-likelihood : -37.8815

Schwarz criterion : 1.4024

Likelihood ratio linearity test : 61.9758

Davies test p-value : 0.0000

Figure 2.1: Evolution of the number of ratings per quarter

We plot total number of rated issuers per quarter. We notice an increase of issuers within time.

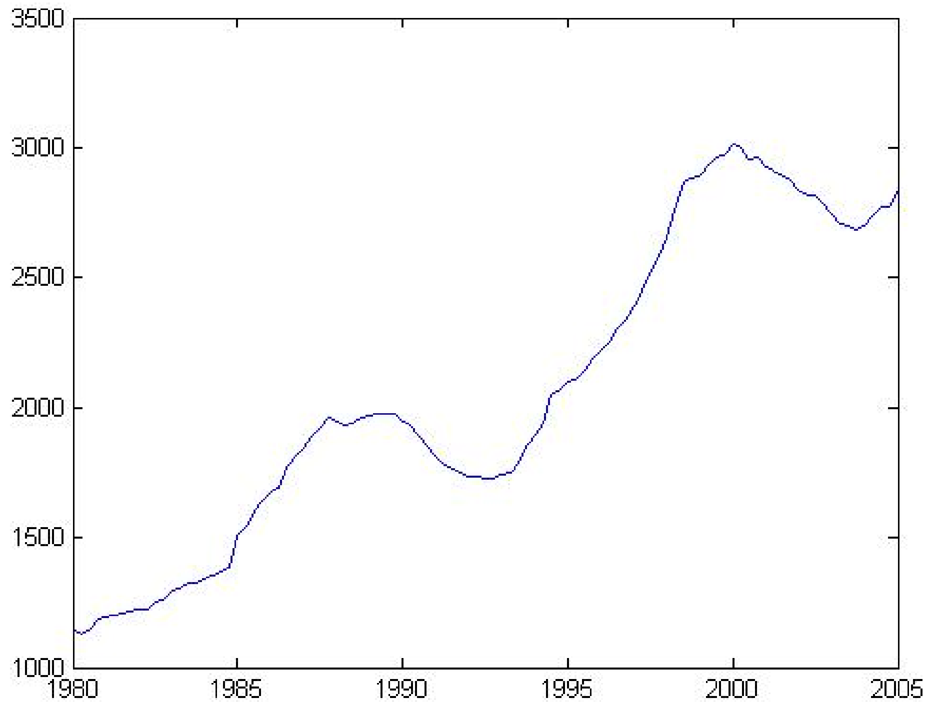


Figure 2.2: Evolution of the number of ratings per quarter for each business sector

For each business sector, we plot total number of ratings per quarter. We observe an increase of rated issuers within time for mainly all sectors.

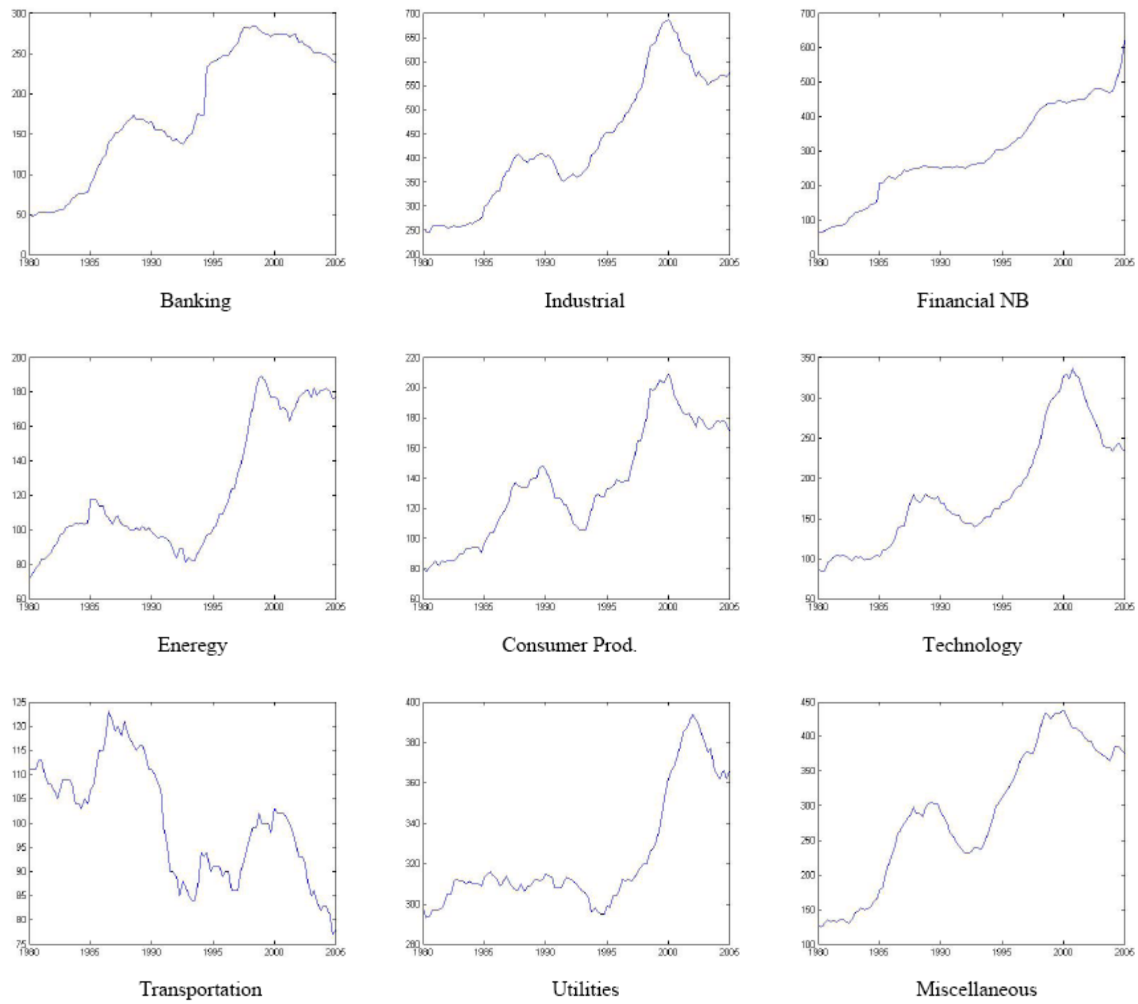


Figure 2.3: Evolution of the mobility metric DC3

We plot the evolution mobility index of each business sector. Extreme peaks and troughs are discussed in the main text.

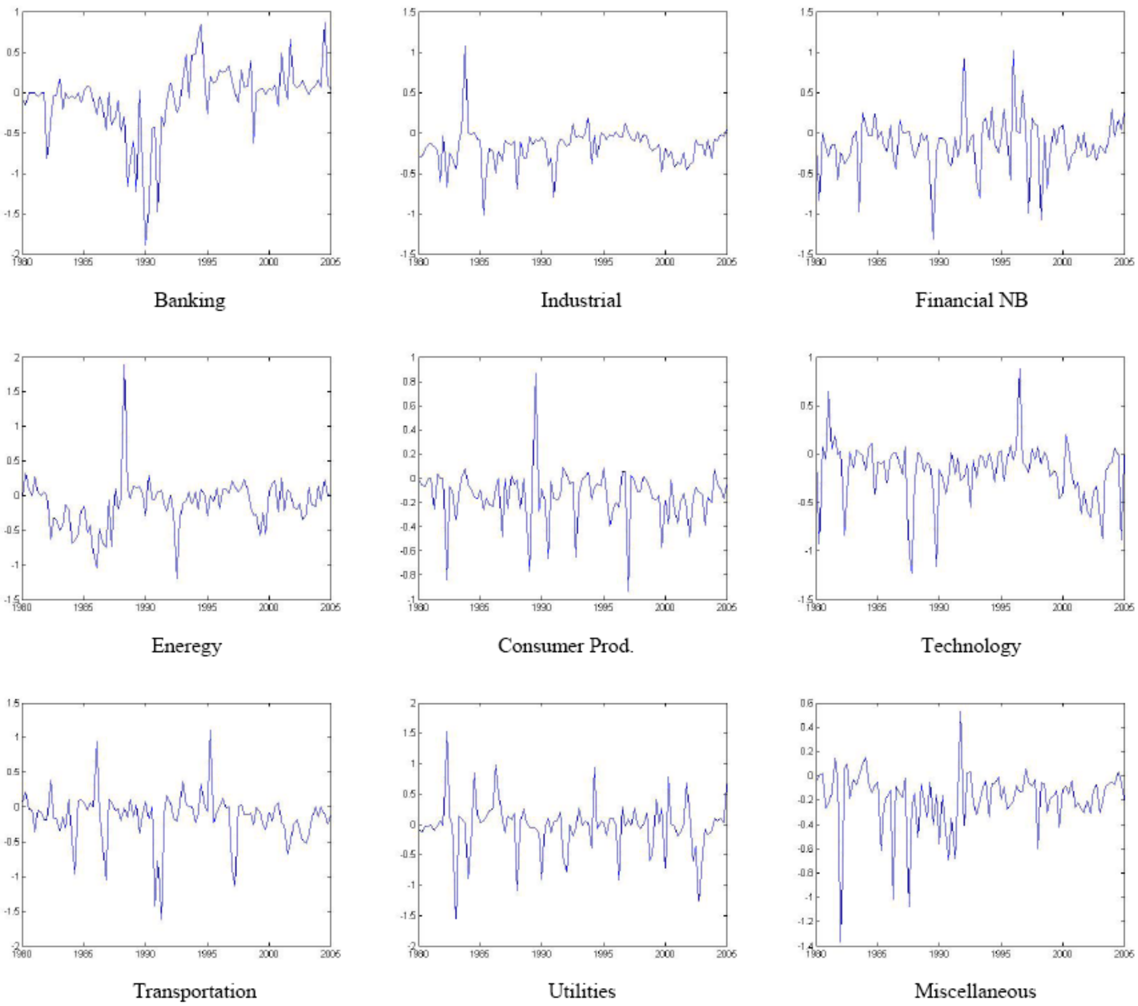
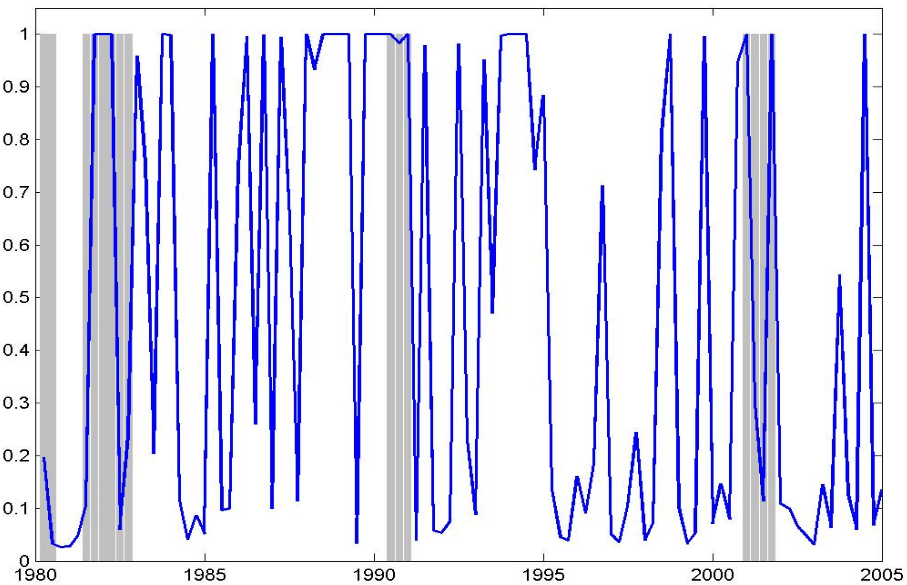


Figure 2.4: Time series of regime probabilities

This figure presents the time series of smoothed probabilities of being in a specific regime. The upper panel presents the results for the low correlation regime (*s*=1). The lower panel corresponds to the high correlation regime (*s*=2). The grey bars indicate contraction periods reported by the NBER.



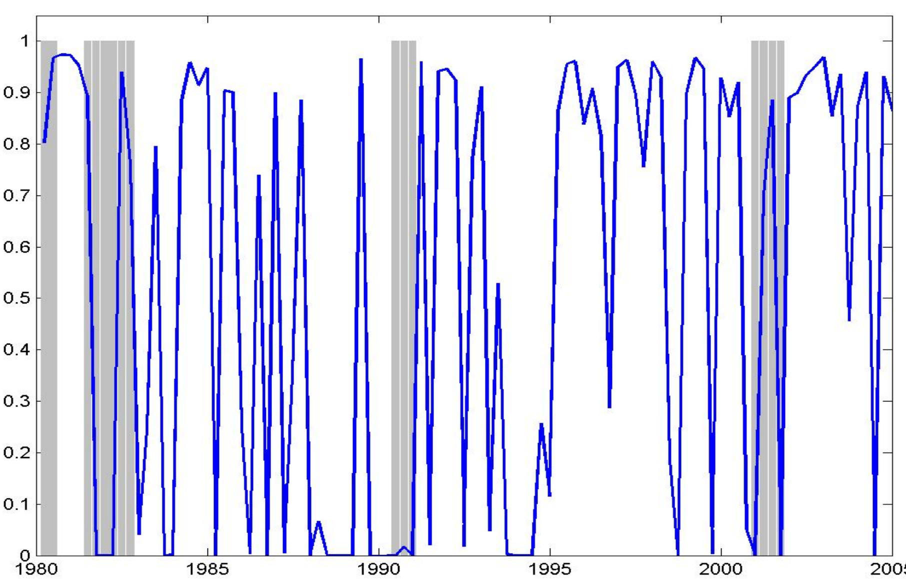


Figure 2.5: Impulse Response Functions for the banking and industrial sectors

The figure indicates the impact of a positive shock in one business sector to the banking and industrial sectors, conditional on being in a regime *s*. 68% confidence intervals are obtained via 500 simulations.

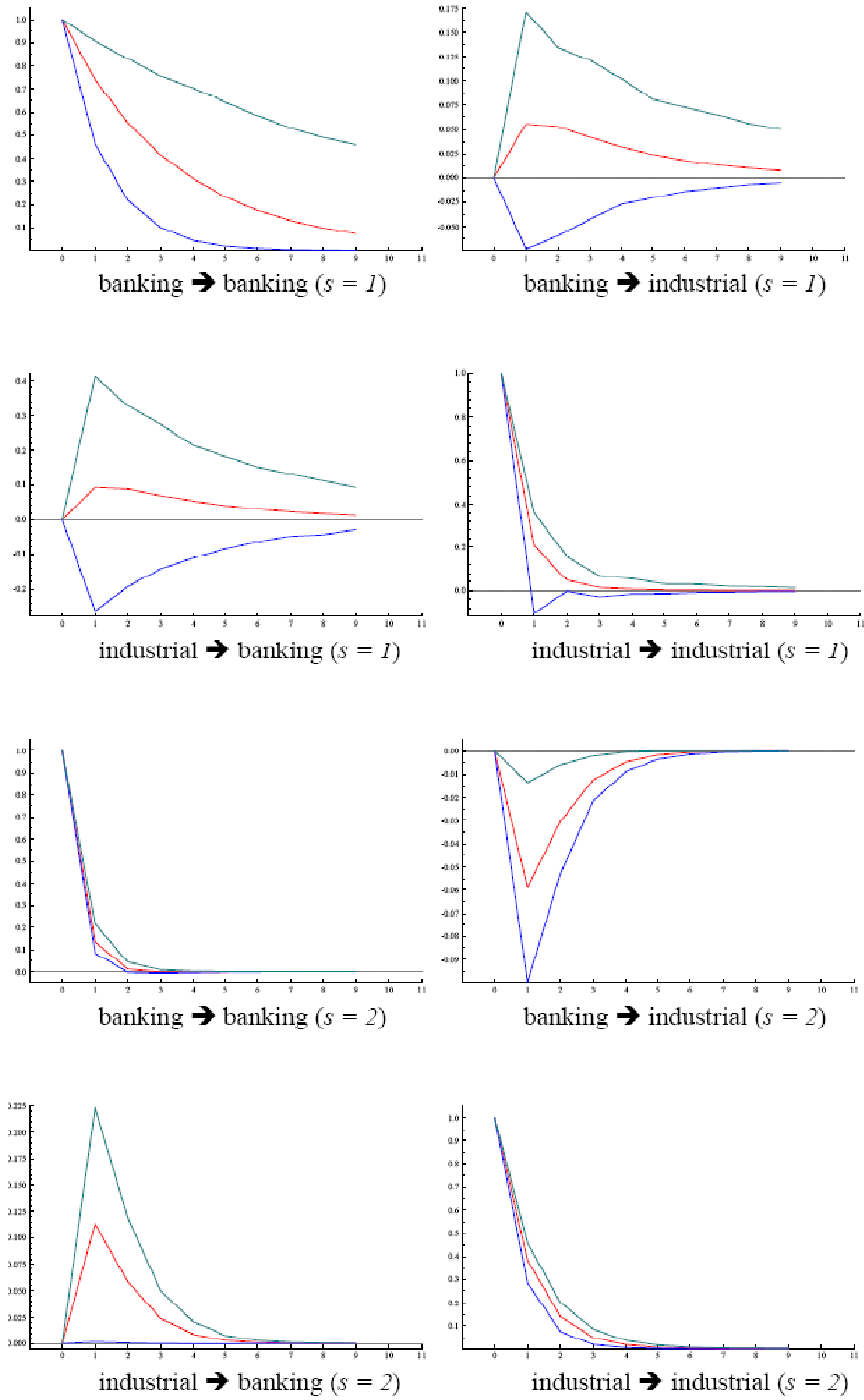


Figure 2.6: Impulse Response Functions for the energy and industrial sectors

The figure indicates the impact of a positive shock in one business sector to the energy and industrial sectors. 68% confidence intervals are obtained via 500 simulations.

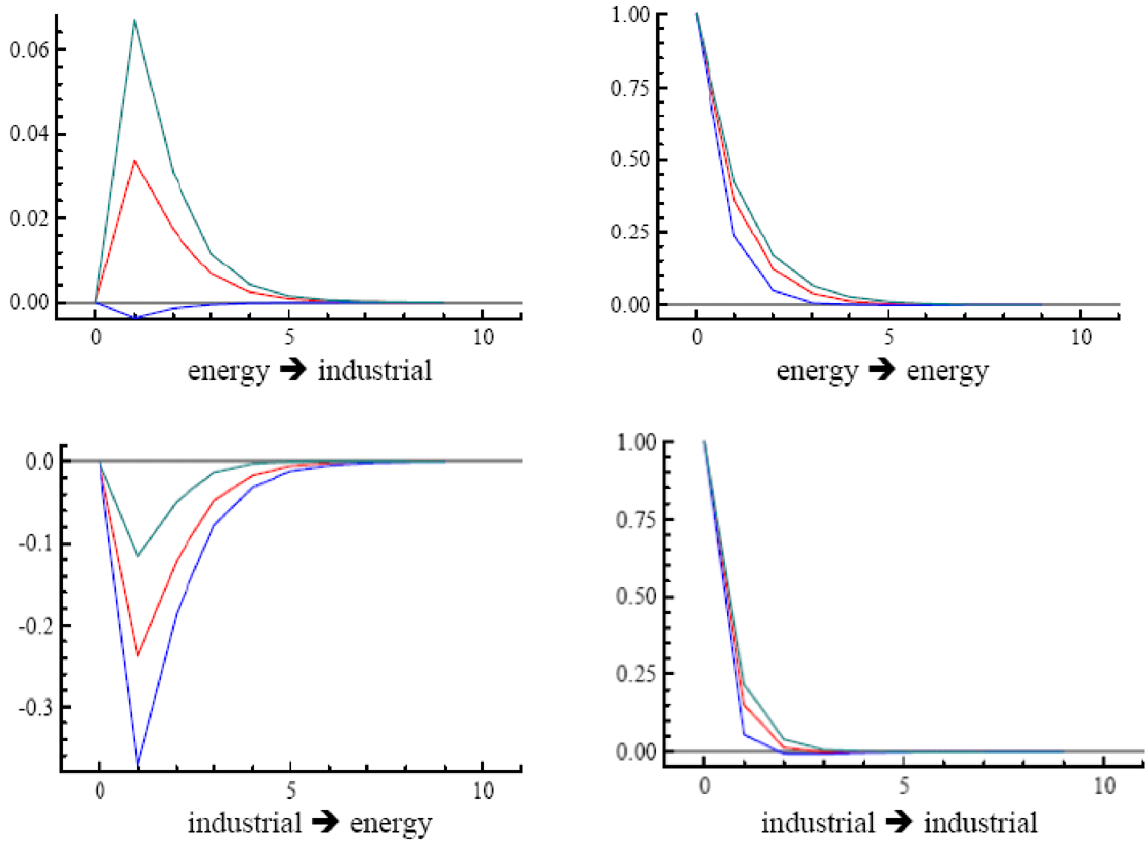
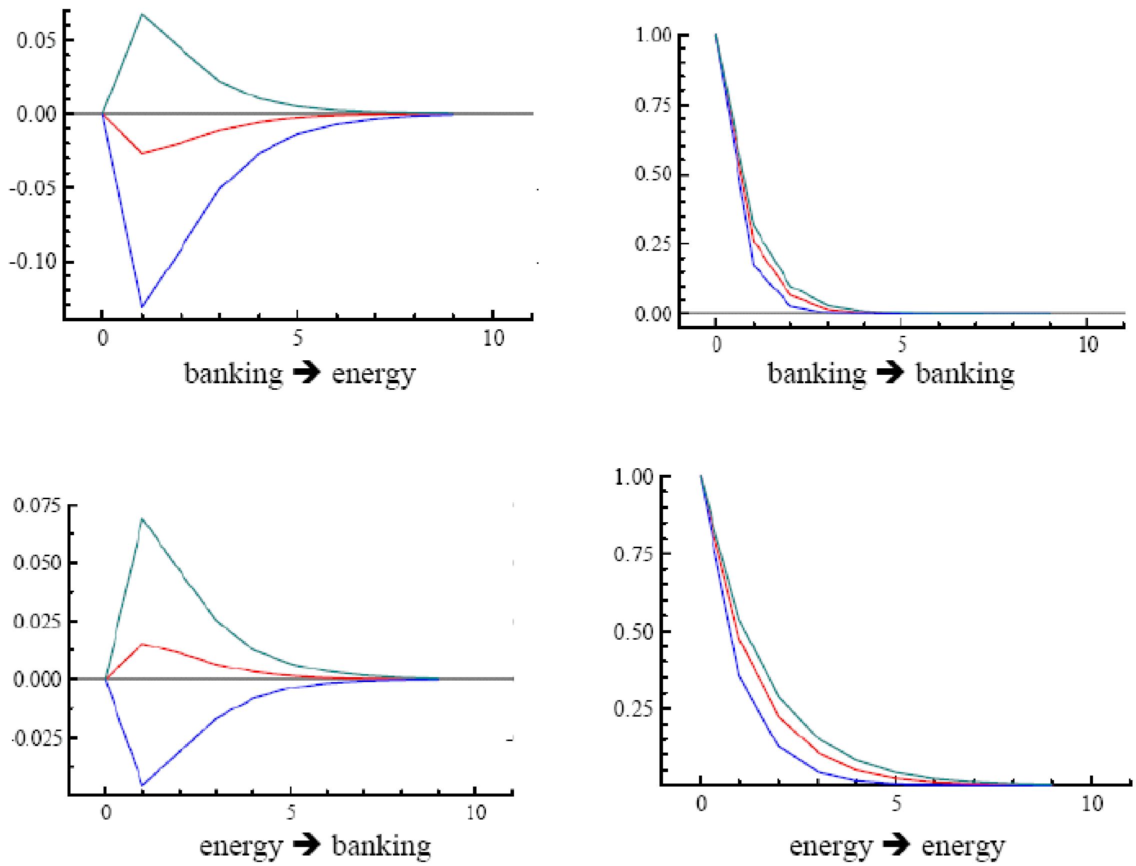


Figure 2.7: Impulse Response Functions for the banking and energy sectors

Figure 2.7 indicates the impact of a positive shock in one business sector to the banking and energy sectors. 68% confidence intervals are obtained via 500 simulations



Chapter 3

Empirical Evaluation of the Asset Allocation Puzzle

3.1 Introduction

This research proposes an empirical solution to the asset-allocation puzzle posed by Canner, Mankiw, and Weil (1997). These authors conclude that the recommendations of some financial advisors are inconsistent with rational allocation as advocated by the modern portfolio theory (MPT). They claim that if, in the presence of a risk-free asset, the bonds/stocks ratio was seen to decrease in relation to risk tolerance (measured by the proportion invested in stocks), this would contradict the conclusion of the two-fund separation theorem which predicts a constant bonds/stocks ratio at all levels of risk tolerance.

Several previous studies have attempted to solve this asset-allocation puzzle by adopting three main lines of research. The first relies on dynamic asset-allocation models: the individual investor tries to maximize his expected utility, while keeping an eye on evolving future returns on the different financial assets (bonds, stocks, and cash). On this topic, we find the works of Bajeux-Besnainou, Jordan and Portait (2001, 2003), of Brennan and Xia (2000, 2002), of Campbell and Viceira (2001), and of Wachter (2003). These studies look at different explanations such as particular specifications of utility function (CRRA, HARA); the link between different financial assets; the inflation factor or the investor's time horizon (finite number of years or infinite horizon).

The second main line of research groups single period theoretical studies. Among these studies, we may cite the contribution of Boyle and Gurthie (2005) who come to the conclusion that the correlation between the return on stocks and human capital could generate a decreasing bonds/stocks ratio, even in a context which authorizes short selling and offers a risk-free asset. We may also cite Elton and Gruber (2000) who have shown that disallowing short-selling and/or eliminating the risk-free asset can explain the bonds/stocks ratio's negative slope with regard to risk tolerance.

Finally, the third main line of research contains empirical studies like the one by Siebenmorgen and Weber (2000) who turn their attention to the asset allocations advocated by German financial advisors. These authors conclude that the choices made by these advisors are rational when viewed through the lens of a behavioural finance model like the one presented by Benartzi and Thaler (2001). We should also cite Shalit and Yitzhaki (2003) who used the same data as Canner et al. (1997) to test investor rationality. Using second-order stochastic dominance as the portfolio optimization criterion, they conclude that the recommendations made by financial advisors were rational.

Our study fits in with the last two lines of research and, more particularly, with the contribution of Elton and Gruber (2000). Our main difference is to use individuals' portfolio choices instead of recommendations from financial advisors. In the second section, we analyze individual investor rationality in a mean-variance single-period framework. We then display the results obtained from an original database of 470 Canadian investor portfolios regarding the relation between bonds/stocks ratios and risk tolerance. Finally, we test for the presence of the separation theorem in order to reach our conclusion on investor rationality.

3.2 Rationality of economic agents

Showing that a bonds/stocks curve which declines in relation to risk tolerance could be consistent with modern portfolio theory, Elton and Gruber (2000) came to the conclusion that the allocations suggested by the financial advisors[1] in Canner et al. (1977) may be rational. This divergence from the conclusions of Canner et al. (1997)

[1] The reference is to Fidelity, Jan Bryant Quinn and Merrill Lynch and to recommendations in the New York Times.

is essentially a function of the context considered: whether a risk free asset is present or not and whether short selling is allowed or not.

The possibility of selling short or not can be cited as one of the rules governing the market. Switching from a context which does authorize short selling to one which does not entails a host of changes. The first consequence is a reduction of the investor's range of possible combinations. The second is related to determining the optimal mean-variance combinations for this same investor. In this respect, it is worth noting that restricting short sales makes the optimal-allocation problem harder to solve. The analytical solution found in a context where short-selling is allowed ceases to be valid when negative proportions of the financial assets are disallowed. In such a case, one alternative means of solving optimal-allocation problems would be through numerical methods.

Obviously, permission to short-sell would be preferable for the reasons cited above. However, current financial market practices should also be kept in view. It is in fact rare to find markets that allow unlimited short-selling by individuals. The cost of short-selling is usually higher. Besides, this practice is not encouraged in many brokerage firms - mainly owing to the extra costs and higher risks short positions entail. Such risks are higher in illiquid markets. Finally, we should point out that Jones and Lamont (2002) and Lamont (2004) have confirmed the existence of regulations banning short sales on some financial markets. This confirmation is based on the observation of several excessively overvalued stocks on these markets.[2] However, it is not obvious that these regulations act to restrict all markets, especially the market of individual investors covered by our study.

We now shift our attention to the hypothesis concerning the existence of a risk-free asset. When the short-selling is allowed, whether or not a risk-free asset exists will not be an important factor in solving the optimal-allocation problem. It is in fact

[2] Jarrow (1980) has shown that regulations banning short-selling imply a price hike in risky assets when all individual investors consider the same variance-covariance matrix.

possible to determine analytically the optimal portfolios for all risk levels. But from a more practical point of view, assumptions concerning the existence of such an asset on financial markets will be less obvious. Though a huge number of corporate and government bonds with nominally constant interest rates do exist on the market, it would be foolhardy to affirm the existence of a totally risk-free asset in the economy. Fluctuating inflation rates cause the yield (in real terms) offered by these bonds to vary over time. However, focusing on a short time horizon might be synonymous with a weak variation in the inflation rate and, consequently, could favour the hypothesis that a risk-free asset does exist.

We need to examine what effect each of these contexts will have on the optimal allocation of assets or, more precisely, on how the bonds/stocks ratio will vary in relation to risk tolerance. Figure 3.1 sums up the cases analyzed by Elton and Gruber (2000). With a risk-free asset and the possibility of short-selling, we should expect a constant bonds/stocks ratio for all investors (no matter what their level of risk tolerance). This ratio becomes a monotone function (either increasing or decreasing) when considering real-term returns (synonymous with the absence of a risk-free asset in an inflationary economy). Restrictions on short-selling will have two possible effects: either a decreasing bonds/stocks ratio in function of risk tolerance or a bonds/stocks ratio which will first increase for relatively low levels of risk tolerance and then later decrease.

Thus, as Elton and Gruber (2000) point out, when the slope of the bonds/stocks ratio is observed to be negative in relation to risk tolerance this should not be understood as a non-optimal investor choice. On the contrary, the theoretical contexts leading to this observation are apparently more in line with practice.

(Figure 3.1 about here)

3.3 Empirical relation between investors' choices and their risk tolerance

3.3.1 Data

In our attempt to solve the asset allocation puzzle posed by Canner et al. (1997), we used data obtained from a Canadian brokerage firm specializing in financial services to individual investors. The originality of this database is that it contains positions chosen by individual investors rather than products offered by brokers as in Canner et al. (1997). These data contain the portfolio composition[3] of 470 of that firm's clients in July 2000, along with their individual characteristics such as age, investment knowledge, income, and investment objectives.

Table 3.1 presents these data.

(Table 3.1 about here)

It appears that 58% of clients claim to have "acceptable" investment knowledge. The percentage of those rating their knowledge as "good" stands at 34%, whereas those claiming "excellent knowledge" represent 4% of the sample. The remaining 4% have no knowledge on the subject. Another aspect which drew our attention concerns the types of accounts held by these individual investors. This datum could, in effect, give us a better idea of each investor's risk aversion. From our observations, we find that all the clients hold a checking account.[4] Of these clients 67% also have a pension fund account; 15% hold a margin account (short-selling); and 2% have both a margin and a pension fund account. Distribution of investors' total net assets is given in Table 3.2.

(Table 3.2 about here)

[3] The portfolio is divided into three classes of assets: Treasury bills, bonds, and stocks.
[4] We should emphasize that these categories are not mutually exclusive. This will be important in the statistical analysis.

One last datum likely to influence asset allocation involves the financial advisor with whom each of the investors deals. In our case, the 470 investors selected use the services of 4 financial advisors. Table 3.3 shows the proportion of clients served by each advisor and provides a brief description of the advisor. Note that the clients of advisors 3 and 4 have been pooled, because these two advisors work together, have the same management style, and the same type of clientele (age, wealth…).

(Table 3.3 about here)

3.3.2 Risk tolerance and portfolio-choice

Drawing on our data, it is easy to construct Figure 3.2 showing the bonds/stocks ratios held by 358 of the 470 clients[5] in terms of their risk tolerance, as measured by the proportion of assets invested in stocks.

(Figure 3.2 about here)

Figure 3.2 shows a negative slope for the bonds/stocks ratio in relation to the proportion of the portfolio invested in stocks. This observation should not, however, imply a confirmation of the paradox mentioned by Canner et al. (1997). Indeed, as a preliminary step in our study, we must check whether the proportion of assets invested in stocks is a good measure of risk tolerance. A second index of risk tolerance "*T(Ind)*" is then calculated for each of the clients, based on their investment objectives:

$$T(Ind) = \frac{1 * inc\% + 2 * growth\% + 3 * spec\%}{3} \quad \text{where} \quad \frac{1}{3} \leq T(Ind) \leq 1. \qquad (3.1)$$

where:

T(Ind): indirect measurement of risk tolerance;

inc%: investment objective in income securities (percentage of total portfolio);

growth%: investment objective in growth securities (percentage of total portfolio);

[5] Of course, we cannot use bonds/stocks ratios for those clients who hold no stocks in their portfolio.

spec%: investment objective in speculative securities (percentage of total portfolio); with *inc% + growth% + spec % = 100%.*

Notice that the average of this second risk-tolerance index is 56% for all the individual investors considered, as compared to an average of 57% for the direct measurement of risk tolerance.

With this indirect measurement of risk tolerance, we can perform the following regressions to test the equivalence between the two measurements:

$$Y = \beta_0 + \beta_1 T(Dir) + \beta_2 Z + \varepsilon_1 \qquad \text{(R3.1)}$$

$$Y = \beta_3 + \beta_4 T(Ind) + \beta_5 Z + \varepsilon_2 \qquad \text{(R3.2)}$$

where:

Y: proportion invested in bonds;[6]

T(Dir): direct measurement of risk tolerance, measured by the proportion invested in stocks;

T(Ind): indirect measurement of risk tolerance;

Z: vector of the individual characteristics of each investor: age, income, size of portfolio, investment knowledge…

The results of these two regressions are presented in Table 3.4 (R3.1 and R3.2).

(Table 3.4 about here)

The equivalence test for the two tolerance measurements (comparison between the parameters $\hat{\beta}_1$ and $\hat{\beta}_4$) indicates that they are not statistically different at a 95%

[6] The proportion invested in bonds is used as a dependent variable in order to include the maximum observations, i.e. 405 clients with all the information needed in the regression. In effect, using the bonds/stocks ratio as the dependent variable would reduce the number of observations by 23% (93 of the 405 clients) because these clients do not hold stocks. When we estimated the model with 312 observations, the results obtained were the same whether based on the proportion invested in bonds or the bonds/stocks ratio. Neither coefficient differs significantly from those presented in Table 3.4. (Details are available upon request.)

confidence level. This observation is later reaffirmed by ensuring that the results obtained are not due to an econometric specification problem.[7] Thus, the proportion invested in stocks serves as a good measurement of risk tolerance and cannot be advanced as a plausible explanation of the paradox posed by Canner et al. (1997).

On the basis of this observation, we propose to analyze the variation in the bonds/stocks ratio in relation to the risk tolerance of each of the 405 clients. Our goal is to explain individuals' asset allocation in terms of their respective risk tolerance and certain other personal variables (age, annual income…). From regression R3.1, it is easy to check whether the bonds/stocks ratio remains constant for all the clients considered. This ratio (designated *r*) can be expressed as follows:

$$r = \frac{Y}{T(Dir)} = \frac{\beta_0 + \beta_1 T(Dir) + \beta_2 Z}{T(Dir)} = \beta_1 + \frac{\beta_0 + \beta_2 Z}{T(Dir)} \qquad (3.2)$$

Thus, the variation of ratio *r* relative to risk tolerance is equal to:

$$\frac{dr}{dT(Dir)} = -\frac{\beta_0 + \beta_2 Z}{\left[T(Dir)\right]^2} \qquad (3.3)$$

Testing whether ratio *r* is constant for all individual investors comes down to testing whether $\hat{\beta}_0$ and $\hat{\beta}_2$ are statistically and jointly equal to zero.[8] The Fisher test rejects this hypothesis at a confidence level of 95% for both regressions (R3.1) and (R3.2) and shows a bonds/stocks ratio which declines in relation to the risk tolerance of the individuals considered (see Table 3.4). In section 3.5, we shall introduce a third measure of risk tolerance to test the robustness of our results.

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[7] For example, when we add $E(T(Dir))$ in (R3.1), the coefficient of $T(Dir)$ becomes –1.05 with a *statistic t* = –96.125. Other results are available upon request.

[8] A Hausman test was performed in order to screen the regression for endogeneity problems.

As Elton and Gruber (2000) note, this rejection of the hypothesis assuming a constant bonds/stocks ratio for all individual investors, based on modern portfolio-choice theory, is not sufficient to conclude for the presence of an asset allocation puzzle. It would be advisable to test also whether the investors considered made their portfolio choices in a context supporting the two-fund separation theorem- thus weighing the rationality of their behaviour.

3.4 Test of the separation theorem

One of the basic hypotheses used in this research consists in accepting the mean-variance model which allows two-fund separation for any increasing and concave utility function, when return distributions belong to the elliptical family.[9] Checking that returns of stocks, bonds, and cash belong to the elliptical family of distributions and testing for the separation theorem will allow us to consolidate our conclusions concerning the rationality of portfolio holders. It is advisable to first present the assets data that the individual investors in our study may have used in their portfolio selection.

3.4.1 Returns on financial assets

To evaluate the returns that individuals in our database considered when making their portfolio selections, we turned to the performance records of three Canadian mutual funds available to the investors of this study. Returns achieved by mutual funds do, in fact, serve as a good indicator for the different financial markets (Barras, Scaillet, and Wermers, 2005).[10]

In evaluating the returns considered by the 470 investors in our sample, we first look at the returns obtained by *Ferique Equity*, *Ferique Bonds*, and *Ferique Short Term Income.*[11] The information available in each of these funds's prospectus will give a

[9] See Owen and Rabinovitch (1983).
[10] Barras et al. (2005), in their survey, based uniquely on data from the U.S., find that just 20% of all equity mutual funds obtain a negative performance.
[11] These returns are available on the *Ferique* funds site (www.ferique.com).

better understanding of our selection. The objective set by *Ferique Equity* is to obtain long-term capital gains by investing in the stocks of Canadian companies. *Ferique Bonds*, for its part, aims to provide a steady stream of high income and, occasionally, some capital gain from investments in Canadian bonds. Its portfolio is composed of Canadian bonds issued by the Canadian government, provinces, municipalities, and corporations. Finally, *Ferique Short Term Income* proposes to provide current income, while protecting capital and maintaining high liquidity. Its portfolio is composed, up to 80%, of Canadian debt securities maturing in under 6 months. Statistics on these returns are presented in Table 3.5.

(Table 3.5 about here)

Returns from the three funds were observed monthly between January 1995 and June 2000. Three remarks justify our selection. First, remember that, for each investor, the portfolio composition considered was that from the month of July 2000. It is thus reasonable to consider returns preceding that date. A second question about these data concerns the frequency with which they were observed. On this point, note that several empirical works[12] on the problem of portfolio selection make use of monthly returns. Finally, an observation period of about 5 years would seem to be a judicious choice. A shorter period will produce less accurate results, whereas estimations based on a very long period run the risk of being affected by changes of regime.

Also worth noting is the strong correlation between returns from the three funds considered and those obtained by certain standard indices over the same period: from January 1995 to June 2000. Indeed, a 91.97% correlation coefficient is observed between the returns generated by *Ferique Equity* and those on the Toronto Stock Exchange index (TSE300). We obtain a 87.24% correlation coefficient between returns on *Ferique Bonds* and those reported by the Scotia Capital (Overall Universe)

[12] According to Elton and Gruber (2000), *"in finance, it is common to use monthly intervals to measure returns used in estimating expected returns, variances and covariances."*

index.[13] Finally, the correlation between returns from *Ferique Short Term Income* and the average return on one-month Treasury bills[14] stands at 80.39%.

Once these returns have been defined historically, we shall then be in a position to see whether they can be considered part of a family of elliptical distributions.

3.4.2 Ellipticality test for returns on financial assets

Two families of tests are generally used to determine the nature of multivariate distributions: the Jarque-Bera (1987) type and the Mardia (1970) type. For an application of the first type, we refer to Kilian and Demiroglu (2000). After first calculating the degree of skewness and kurtosis for each separate random variable, an aggregation of these univariate results produces statistics related to the multivariate distribution. Tests of the Mardia type allow a direct calculation of the multivariate skewness as well as the multivariate kurtosis. These tests have the advantage of taking into account the correlation between the different random variables in the joint distribution. These tests also lead to the same results as those obtained with the Jarque-Bera type test in the univariate case and seem more suitable to the multivariate case.

A Mardia-type test is based on two statistics: the multivariate skewness (*MSK*) and the multivariate kurtosis (*MKU*). First, we present these two statistics. We then describe the methodology adopted to determine the nature of the joint distribution of the returns on stocks, on bonds, and on cash.

Let N risky assets be observed over T periods; we have T vectors $R_1, R_2, \ldots, R_T$; each vector contains the returns observed at a given date for the N risky assets considered. We can note by d_{ts} all elements of the matrix $\left(R_t - \overline{R}\right)' S^{-1}\left(R_s - \overline{R}\right)$ for all t and s

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[13] Returns related to the TSE300 Index and the Scotia Capital Index were drawn from the Datastream base.
[14] This datum was obtained from the Bank of Canada site: identifier V122529 in the CANSIM directory.

contained between 1 and T where $\overline{R}$ is the vector of average returns and S the variance-covariance matrix of the same returns. The multivariate skewness and kurtosis are calculated as follows:

$$MSK = \frac{1}{T^2}\sum_{t=1}^{T}\sum_{s=1}^{T} d_{ts} \quad \text{and} \quad MKU = \frac{1}{T}\sum_{t=1}^{T} d_{tt}^2 \tag{3.4}$$

These two statistics will serve as the basis for determining the distribution of the returns observed for stocks, cash and bonds. The first step consists in calculating statistics MSK and MKU (noted respectively as MSK_{obs} and MKU_{obs}) relative to the series of returns observed. The second step is based on simulations. We make T drawings of N random variables according to a precisely determined distribution (multivariate Student, multivariate normal, …), taking into account a variance-covariance matrix equivalent to the one linked to the observations. Based on these simulated data, it is possible to calculate statistics MSK and MKU. The results obtained will be noted as $MSK_{sim,1}$ and $MKU_{sim,1}$. Repeating these simulations M times will allow us to obtain the following vectors: MSK_{sim} = [$MSK_{sim,1}$, $MSK_{sim,2}$, …, $MSK_{sim,M}$]' and MKU_{sim} = [$MKU_{sim,1}$, $MKU_{sim,2}$, …, $MKU_{sim,M}$]'. We next classify the elements of these two vectors to find the vectors MSK_{sim}^{ord} and MKU_{sim}^{ord}. Finally, noting the fact that MSK_{obs} is bounded by the $\left[\frac{\alpha}{2}M+1\right]^{th}$ value and the $\left[\left(1-\frac{\alpha}{2}\right)M-1\right]^{th}$ value of vector MSK_{sim}^{ord} and that MKU_{obs} is bounded by the $\left[\frac{\alpha}{2}M+1\right]^{th}$ value and the $\left[\left(1-\frac{\alpha}{2}\right)M-1\right]^{th}$ value of vector MKU_{sim}^{ord}, this allows us to conclude that the returns observed follow the multivariate distribution simulated at a confidence level of $(1-\alpha)$.

We shall now apply the methodology described above to our data: monthly returns noted between January 1995 and June 2000 for *Ferique Equity, Ferique Bonds,* and

Ferique Short Term Income. The statistics related to the returns observed stand at 2.2493 and 15.7729 respectively for the multivariate skewness and kurtosis. These values lead us to conclude, at a 99% confidence level, that the returns observed do not reject the multivariate Student distribution.[15] For instance by simulating a multivariate Student distribution with 10 degrees of freedom, the confidence interval related to the multivariate skewness is equal to [0.2620; 8.3508], whereas that related to the multivariate kurtosis corresponds to [12.8882; 29.3019].

Finding that the returns on stocks, bonds, and cash may correspond to one of the elliptical distributions (Student distribution with 10 degrees of freedom) allows us to test the Black-CAPM by assuming there is no risk free asset and no restriction on short selling - both reasonable assumptions for the observed investment environment of our initial data set.

3.4.3 Black-CAPM test with non-gaussian returns

Beaulieu, Dufour, and Khalaf (2003) have presented a test of the Black's Capital Asset Pricing Model (BCAPM) with possibly non-gaussian returns. It seems advisable to adopt their methodology, since we are dealing with three risky assets whose returns seem to correspond to a Student distribution. We shall now present the methodology used to test the Black-CAPM (BCAPM). A brief introduction to the model is required to explain the notations to be used.

Note as R_{it}, $i = 1, \ldots, n$, the returns on n risky assets during period t (stretching from 1 to T) and as $\tilde{R}_{Mt}$ the returns on the market portfolio. The BCAPM test will thus be based on the following model:

$$R_{it} = a_i + b_i \tilde{R}_{Mt} + u_{it}; \; t = 1, \ldots, T, \; i = 1, \ldots, n \qquad (3.5)$$

[15] These results were obtained based on 9999 simulations.

where u_{it} designates the error term. In fact, testing BCAPM comes down to checking whether there is a scalar γ (return on the zero-beta portfolio whose composition is unknown to us) such that:

$$H_{BCAPM}\text{: } a_i = \gamma(1 - b_i), \ \forall i = 1,...,n \qquad (3.6)$$

Model (3.5) can be re-written under the matricial form:

$$Y = XB + U \qquad (3.7)$$

with:

$$Y = [R_1, \ldots, R_n], \ X = [\iota_T, \tilde{R}_M];$$

$$R_i = (R_{1i}, \ldots, R_{Ti})' ; \ \tilde{R}_M = (\tilde{R}_{1M}, \ldots, \tilde{R}_{TM})' \text{ and } \iota_T = (1, \ldots, 1)' .$$

Finally, the hypothesis test presented in (3.6) is based on the calculation of the quasi likelihood ratio calculated as follows:

$$LR_{BCAPM} = T \ln(\Lambda_{BCAPM}) \text{ with } \Lambda_{BCAPM} = \left|\hat{\Sigma}_{BCAPM}\right| \big/ \left|\hat{\Sigma}\right| \qquad (3.8)$$

where:

$$\hat{\Sigma} = \hat{U}'\hat{U} / T \ ; \ \hat{U} = Y - X\hat{B} \text{ and } \hat{B} = (X'X)^{-1} X'Y ,$$

and $\hat{\Sigma}_{BCAPM}$ designates the $\hat{\Sigma}$ estimator in the constrained model which verifies hypothesis (3.6).

One of the basic hypotheses of the methodology of Beaulieu et al. (2003) is the possibility of re-writing vector $U_t = (u_{1t}, \ldots, u_{nt})'$ as the product of an unknown triangular matrix J and a vector $W_t = (W_{1t}, \ldots, W_{nt})'$ whose joint distribution is fully specified. We thus obtain the following equalities:

$$U_t = JW_t \tag{3.9}$$

$$\Sigma = JJ' \tag{3.10}$$

where Σ designates the variance-covariance matrix of vector U_t.

Given this hypothesis advanced by Beaulieu et al. (2003), the likelihood ratio, defined by expression (3.8), is distributed as follows:

$$LR(\gamma_0) = T \ln\left(|W' M_0 W| / |W' M W|\right) \tag{3.11}$$

where:

$$M = I - X(X'X)^{-1} X'$$

and

$$M_0 = M + X(X'X)^{-1} H' \left[H(X'X)^{-1} H'\right]^{-1} H(X'X)^{-1} X'$$

with $H = [1 \;\; \gamma_0]$ and $W = [W_1, \ldots, W_T]'$.

The BCAPM test thus comes down to first setting a value for the scalar γ_0 and then calculating the likelihood ratio it entails. The second step consists in simulating N drawings for the multivariate distribution W. For each of these drawings, we calculate the likelihood ratio as defined by expression (3.11). Calculation of the specific *p-value* of the scalar γ_0 is obtained as follows:

$$\hat{p}_N\left(LR(\gamma_0) \mid \nu\right) = \frac{\hat{G}_N(\gamma_0, \nu) + 1}{N + 1} \tag{3.12}$$

where v designates the parameters of the distribution used during the simulations (such as the degree of freedom during simulation of a Student distribution) and $\hat{G}_N(\gamma_0, \nu)$ corresponds to the number of ratios resulting from the simulations which exceed the ratio calculated based on the observations.

This calculation of *p-values* is repeated for several possible values of ν and the *p-value* of the BCAPM test is obtained as follows:

$$\hat{p}_N^*(LR(\gamma_0)) = \sup_{\nu} \hat{p}_N(LR(\gamma_0)|\nu) \qquad (3.13)$$

In the end, the decision rule concerning the hypothesis test cited in (3.6) consists in comparing the *p-value* transferred to (3.13) and the level of significance α considered: if the *p-value* exceeds α, the BCAPM hypothesis is not rejected.

We now apply this BCAPM test to our particular context. Considering *Ferique Balanced* as a market portfolio[16] based on the results related to the distribution of returns from *Ferique Equity*, *Ferique Bonds*, and *Ferique Short-Term Income*, we reach the conclusion that the BCAPM is not rejected at a 99% level of confidence. (see Table 3.6 for a detailed presentation of the empirical results).

(Table 3.6 about here)

To consolidate our conclusions, certain robustness tests are advisable. Indeed, our previously results might depend on approximations of risk tolerance and returns from stocks, bonds, cash and the market portfolio. These results could also arise from the methodology used to test the BCAPM.

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[16] The monthly returns between January 1995 and June 2000 are considered. The portfolio of this fund is composed of stocks, bonds and short-term assets. Statistics of these returns are available in Table 3.5.

3.5 Robustness of results

Our first robustness test concerns the measurement of risk tolerance. The results obtained by the indirect measurement of risk tolerance defined in (3.1) might in fact depend on the coefficients assigned. Erroneous interpretation of these coefficients may occur: it may be arbitrary to suppose that an investor placing his money in speculative assets is 3 times more risk tolerant than the one who places his money in income assets.

We thus propose a third risk-tolerance measurement which is defined as follows:

$$T(Obs) = \frac{rmon * mon\% + rbond * bond\% + rstc * stc\%}{rstc} \tag{3.14}$$

where *mon%, bond%* and *stc%* represent respectively the proportions each investor holds in money, bonds, and stocks. *rmon, rbond* and *rstc* designate the average returns on the money, bonds, and stocks considered by all the investors.[17]

A more risk-tolerant investor will tend to place his wealth in high-risk financial assets, which are synonymous with higher returns. Thus an investor’s observed portfolio returns should be indicative of his risk tolerance. The regression of the proportion invested in bonds in relation to this new risk-tolerance measurement is presented in Table 3.4 (R3.3). Defined by average returns on the different financial assets and by the portfolio’s composition, this composite measurement also indicates a negative slope of the bonds/stocks ratio relative to risk tolerance, with a 95% confidence level.

Our second robustness test consists in using other approximations to calculate returns from stocks, bonds, liquidities, and the market portfolio. As an alternative to *Ferique*

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[17] In our case, these average monthly returns amount to 0.4%, 0.7%, and 1.34% for money, bonds and stocks respectively, supposing that individual investors turn to *Ferique* funds in making their portfolio selections.

funds, we use the returns generated by *Talvest* funds or by *TD* funds between January 1995 and June 2000.[18]

To evaluate returns on stocks, we selected *Canadian Equity Value* from the *Talvest* funds. As indicated in its prospectus, the fund's objective is to obtain higher than average long-term capital growth, by investing mainly in Canadian equity securities. The bond yield is evaluated by the performance of the *Talvest Bond Fund*. This fund's objective is to maintain capital while obtaining high current income, by investing mainly in bonds, debentures, notes, and other debt instruments of financial institutions, corporations, and Canadian governments. Approximation of the return on cash is based on returns generated by the *Talvest Money Market Fund.* This fund proposes to obtain high income, while protecting both capital and liquidity, by investing mainly in high-quality, short-term debt securities issued or guaranteed by the government of Canada or by one of its provinces. Descriptive statistics of the returns of these funds are presented in Table 3.5.

We should notice the high correlations between *Talvest* funds and other funds having the same objectives. For example, we obtain a 84.14% correlation coefficient between returns on *Ferique Equity* and *Canadian Equity Value* from *Talvest* funds. Other correlations between returns on the funds considered are available in Table 3.7.

(Table 3.7 about here)

The first test applied to the returns on these three funds does not permit us to reject the null hypothesis of elliptically distributed returns at the 99% confidence level. The multivariate skewness in the joint distribution of these returns actually amounts to 1.7369, whereas the multivariate kurtosis is equal to 17.9584, These statistics range

[18] Direct access to the performance of the funds selected are available on the two following Web sites: www.talvest.com and www.tdcanadatrust.com.

within the intervals at the 99% confidence level for the multivariate Student distribution with 15 degrees of freedom[19], based on 9999 simulations.

This non-rejection of the ellipticity of the returns allows us to test for the Black's Capital Asset Pricing Model (BCAPM), applying the methodology used in section 3.4. The market portfolio considered in this test corresponds to *Talvest's Canadian Asset Allocation Fund* whose stated objective is to obtain long-term stable capital growth, by investing mainly in a balanced portfolio composed of Canadian equity and debt securities, including money market instruments. The BCAPM test on *Talvest* funds, and on 9999 simulations of the multivariate Student distribution, does not reject the null hypothesis of the presence of the separation theorem at the 99% confidence level (details are in Table 3.6).

The methodology cited above was also applied to *TD* funds. *TD Canadian Money Market Fund*, *TD Canadian Equity Fund*, and *TD Canadian Bond Fund* were selected to evaluate, respectively, the return on cash, stocks, and bonds traded in Canada.

The evaluation based on these data also prevents us from rejecting the elliptical distribution of returns at the 99% confidence level. Indeed, we obtain a skewness equal to 2.0406 and a kurtosis of 17.1734 for the joint distribution, whereas the intervals of confidence obtained for a multivariate Student distribution with 8 degrees of freedom are [0.2766; 12.0547] and [13.0778; 33.8362] respectively.

This observation leads us to test the BCAPM based on the *TD Balanced Fund* as a market portfolio. Applying the method cited above and based on our 9999 simulations, we do attain the non-rejection of the BCAPM at the 99% confidence level.

[19] The confidence interval for the multivariate skewness is [0.2158; 4.6242], whereas that for the multivariate kurtosis is [12.5134; 23.0327] at the 99% confidence level and for the multivariate Student distribution at 15 degrees of freedom.

It would also be advisable to apply a second methodology for testing the BCAPM in the presence of non-Gaussian returns. This second technique, drawn from the work of Zhou (1993), differs from that of Beaulieu et al. (2003) by its non-separation between the nuisance terms (the unknown triangular matrix *J*) and the *W* vector whose joint distribution is fully specified. In adopting this second methodology, the distribution of the likelihood ratio will be:

$$LR(\gamma_0) = T \ln\left(\left| \hat{U}' M_0 \hat{U} \right| \Big/ \left| \hat{U}' M \hat{U} \right| \right) \qquad (3.15)$$

where $\hat{U}$, M_0 and M are as defined above.

To obtain Zhou's model estimates, it thus suffices to apply the methodology proposed by Beaulieu et al. (2003) in making separate draws based on a *U* distribution instead of the fully specified *W* distribution.

When applied to our three families of funds (*Ferique*, *Talvest* and *TD*), this methodology does not allow us to reject the null hypothesis for the existence of the separation theorem at the 99% confidence level. For each of the three tests, we had recourse to 9999 simulations based on multivariate Student distributions. (See panel Zhou (1993) test in Table 3.6 for a detailed presentation of the empirical results.)

3.6 Conclusion

We have provided new elements of response to the asset-allocation puzzle posed by Canner et al. (1997). We first present a careful verification of the reliability of the risk-tolerance measurement used by the authors and obtained a positive result. Our methodology for evaluating the rationality of the portfolios choices made by individual investors is based on Elton and Gruber (2000) study. We therefore tested for the existence of the separation theorem, based on data reflecting all possible portfolio selections for investors with a negative empirical relation between the bonds/stocks ratio and the risk-tolerance index. This test is carried out in two stages:

the first consists in checking for the ellipticality of the returns observed on the market, whereas the second involves a test of the Black's CAPM with possibly non-Gaussian error terms and assuming an environment with no risk free asset and unrestricted short selling. The results obtained favour elliptical returns and confirm the Black-CAPM hypothesis. We then conclude that the asset allocation puzzle does not exist in our data set.

Finally, our results very likely prove that the investors in our data base are not squeezed by constraints on short selling. The fact that very few of them hold negative proportions of assets is more a reflection of personal choice than of a tight constraint.

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Table 3.1: Descriptive statistics of data

Table 3.1 displays descriptive data of 470 clients of the brokerage firm in July 2000 such as the average age, income, amount in portfolio, investment objectives, …

	Average	**Standard Deviation**
Age	54	13
Income ($)	48 034	52 390
Amount in portfolio ($)	91 857	153 494
Income assets objective	42%	31%
Growth assets objective	49%	29%
Speculative assets objective	9%	18%
Weight in stocks	57%	38%
Weight in bonds	38%	40%
Weight in liquidities	5%	7%
Bonds/stocks ratio	0.77	2.52
Excellent knowledge	4%	20%
Good knowledge	34%	47%
Acceptable knowledge	58%	49%
No knowledge	4%	20%
Pension fund account	67%	47%
Margin account	15%	36%
Margin account and Pension fund account	2%	14%

Table 3.2: Distribution of total net assets of the investors

This table indicates the distribution of the total net assets of the 470 clients of our sample.

Total net assets (in $ 1000)	Less than 25	25 to 50	50 to 100	100 to 250	250 to 500	More than 500
Proportion	4%	5%	13%	37%	27%	14%

Table 3.3: Distribution of clients according to financial advisors

This table indicates the distribution of the 470 clients of our sample according to the 4 financial advisors. A brief description of each advisor is also included.

	Proportion	**Description**
Advisor 1	27%	Young advisor, 3 years of experience
Advisor 2	54%	10 years of experience, with a large clientele
Advisors 3 and 4	19%	15 years of experience, with a wealthy clientele

Table 3.4: Results from regressions using different measurements of risk tolerance

Table 3.4 reports the results of regressions (R3.1) (R3.2) and (R3.3). The total number of observations stands at 405 for each regression. *t*-statistics of estimated coefficients are reported in parentheses. The following dummy variables are part of the constant: Financial advisor 2, Good knowledge, Margin account **and** Pension fund account, and Asset (100 to 250). The slope *F* statistic allows us to test if $\hat{\beta}_0$ and $\hat{\beta}_2$ are jointly and significantly different from zero. The tabulated Fisher statistic $F(16,388)$ at a confidence level of 95% stands at 1.6696.

Independent Variables	Direct Measurement (R3.1)	Indirect Measurement (R3.2)	Indirect Measurement (R3.3)
Constant	0.9846**	0.7761**	2.1340**
	(43.4066)	*(4.9701)*	*(49.0278)*
T(Dir)	-1.0587**		
	(-105.3345)		
T(Ind)		-0.8896**	
		(-5.7272)	
T(Obs)			-2.2397**
			(-67.8738)
Age	0.0003	0.0045**	0.0006
	(0.9202)	*(2.6350)*	*(1.3839)*
Income ($1000)	0.0001	-0.0006	0.0001
	(1.0587)	*(-1.2839)*	*(0.8311)*
Asset (0 to 25)	-0.0022	0.0868	-0.0015
	(-0.1227)	*(0.9273)*	*(-0.0563)*
Asset (25 to 50)	-0.0100	0.0450	-0.0127
	(-0.6572)	*(0.5663)*	*(-0.5515)*
Asset (50 to 100)	0.0014	0.0591	0.0034
	(0.1200)	*(0.9899)*	*(0.1968)*

Table 3.4 continued

Asset (250 to 500)	0.0011	0.0124	0.0010
	(0.1226)	*(0.2692)*	*(0.0747)*
Asset (more than 500)	0.0031	0.0941	0.0068
	(0.2540)	*(1.4908)*	*(0.3696)*
Amount of portfolio ($1000)	-0.0001**	-0.0006**	-0.0002**
	(-2.9735)	*(-3.4170)*	*(-3.2080)*
Financial advisor 1	-0.0037	-0.1724**	-0.0130
	(-0.4145)	*(-3.6130)*	*(-0.9695)*
Financial advisors 3&4	-0.0091	-0.0188	-0.0151
	(-0.9051)	*(-0.3572)*	*(-0.9905)*
Pension fund account	-0.0199*	0.0154	-0.0293*
	(-2.5246)	*(0.3736)*	*(-2.4509)*
Margin account	0.0206	-0.1037	0.0273
	(1.9192)	*(-1.8626)*	*(1.6763)*
Excellent knowledge	0.0033	-0.2019*	-0.0002
	(0.1818)	*(-2.1493)*	*(-0.0058)*
Acceptable knowledge	0.0034	-0.0605	0.0033
	(0.4540)	*(-1.5675)*	*(0.2911)*
No knowledge	-0.0151	-0.0363	-0.0230
	(-0.8407)	*(-0.3864)*	*(-0.8442)*
	R^2 = 97.40%	**R^2 = 28.80%**	**R^2 = 94%**
Regression *F* statistic (16, 388)	904.52	9.79	379.74
Slope *F* statistic (16, 388)	1660.52	12.24	395.43

* significant at 5%;

** significant at 1%

Table 3.5: Statistics on returns used in our separation test

Table 3.5 reports descriptive statistics on returns from Ferique funds, Talvest funds and TD funds.

	Average	**Standard Deviation**	**Skewness**	**Kurtosis**
Ferique – Equities	0.0134	0.0395	- 0.9924	5.9434
Ferique – Bonds	0.0070	0.0133	0.3540	2.7708
Ferique – Money	0.0040	0.0011	0.8953	3.9003
Ferique – Balanced	0.0117	0.0282	- 0.8186	5.7848
Talvest – Equities	0.0137	0.0483	- 0.6383	7.5227
Talvest – Bonds	0.0065	0.0141	0.3682	3.2926
Talvest – Money	0.0036	0.0011	0.7644	3.3844
Talvest – Balanced	0.0101	0.0302	- 0.7545	5.1402
TD – Equities	0.0146	0.0458	- 1.0841	7.3410
TD – Bonds	0.0094	0.0158	0.2713	2.7007
TD – Money	0.0036	0.0009	0.6045	2.7264
TD – Balanced	0.0096	0.0234	- 1.3929	7.8637

Table 3.6: Results of the BCAPM test for the various funds studied

The results of the BCAPM test are provided in table 3.6. Two methodologies are used, namely the Beaulieu et al.(2003) test and the Zhou (1993) test.

	Beaulieu et al. (2003) test			**Zhou (1993) test**		
	LR	γ_0	*p-value*	LR	γ_0	*p-value*
Ferique Funds	4.756	0.373%	0.224	4.756	0.373%	0.228
Talvest Funds	5.278	0.330%	0.174	5.278	0.330%	0.176
TD Funds	5.228	0.361%	0.186	5.228	0.361%	0.184

LR: Likelihood ratio

p-value: When *p-value* is greater than 0.01, we do not reject the Black-CAPM at 99% level of confidence.

γ_0 : Return of the zero covariance portfolio

Table 3.7: Correlation between returns on the funds considered

Table 3.7 reports the correlation between returns on the 12 funds studied.

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
Ferique – Equities (1)	100	49.17	12.03	92.99	84.14	35.91	0.77	88.93	87.24	31.94	2.20	87.58
Ferique – Bonds (2)	–	100	22.46	61.34	46.14	87.77	12.72	50.61	40.54	84.43	17.61	59.11
Ferique – Money (3)	–	–	100	10.15	8.49	14.56	85.91	9.09	7.21	17.15	87.81	14.84
Ferique – Balanced (4)	–	–	–	100	82.92	46.93	-3.97	92.62	85.30	43.71	-2.17	88.29
Talvest – Equities (5)	–	–	–	–	100	40.74	-0.54	89.24	90.98	41.51	1.22	80.12
Talvest – Bonds (6)	–	–	–	–	–	100	15.32	46.38	31.65	96.00	18.23	55.83
Talvest – Money (7)	–	–	–	–	–	–	100	-0.30	-4.53	17.27	92.8	4.89
Talvest – Balanced (8)	–	–	–	–	–	–	–	100	90.20	44.57	-0.32	89.27
TD – Equities (9)	–	–	–	–	–	–	–	–	100	32.40	-4.06	82.62
TD – Bonds (10)	–	–	–	–	–	–	–	–	–	100	19.39	53.35
TD – Money (11)	–	–	–	–	–	–	–	–	–	–	100	5.55
TD – Balanced (12)	–	–	–	–	–	–	–	–	–	–	–	100

Figure 3.1: Short-selling and risk free asset: Impact on optimal asset allocation

In this figure, we display the impact of allowing short-selling and a presence of a risk free asset on optimal asset allocation. The 4 possibles cases are studied.

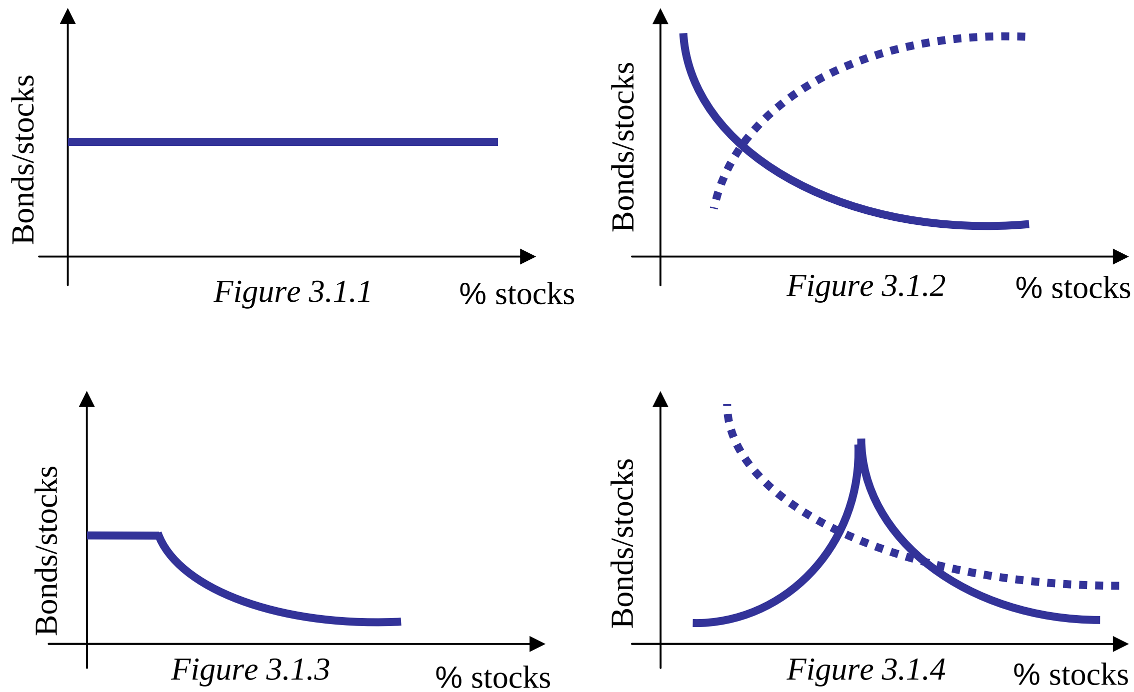

Figure 3.1.1 Possibility of short-selling and presence of risk- free asset
Figure 3.1.2 Possibility of short-selling and absence of risk- free asset
Figure 3.1.3 No short-selling and presence of risk- free asset
Figure 3.1.4 No short-selling and absence of risk- free asset

Figure 3.2: Variation of bonds/stocks ratio observed, as based on proportion of portfolio invested in stocks

Figure 3.2 reports the bonds/stocks ratios observed from our sample, as based on the risk tolerance (proxied by the proportion of portfolio invested in stocks*)*