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Three Essays on High Frequency Financial Data and Their Use for Risk Management

by

Maria Pacurar

Finance Department and Canada Research Chair in Risk Management
HEC Montréal

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Has been evaluated by the following jury:

President-reviewer:	Dr Joshua Slive
	HEC Montréal
Thesis supervisor:	Dr Georges Dionne
	HEC Montréal
Thesis co-supervisor:	Dr Pierre Duchesne
	Université de Montréal
Internal examiner:	Dr Jean-Guy Simonato
	HEC Montréal
External examiner:	Dr Christian Gouriéroux
	Université Paris IX et ENSAE

Abstract

This dissertation contains three essays investigating the modeling and use of financial tick-by-tick data. High-frequency finance has become a very active field of research over the last two decades. Research making use of irregularly time-spaced transaction data has its roots in the seminal article of Engle and Russell (1998) that introduced the Autoregressive Conditional Duration (ACD) model for the analysis of arrival times between events based on all past information.

The first essay provides an up-to-date survey of the main theoretical developments in ACD modeling and empirical studies using financial data. First, we discuss the properties of the standard ACD specification and its extensions, existing diagnostic tests, and joint models for the arrival times of events and some market characteristics. Then, we present the empirical applications of ACD models to different types of events, and identify possible directions for future research.

The second essay proposes two classes of test statistics for duration clustering and one class of test statistics for the adequacy of ACD models, using a spectral approach. The tests for ACD effects of the first class are obtained by comparing a kernel-based normalized spectral density estimator and the normalized spectral density under the null hypothesis of no ACD effects, using a norm. The second class of test statistics for ACD effects exploits the one-sided nature of the alternative hypothesis. The class of tests for the adequacy of an ACD model is obtained by comparing a kernel-based spectral density estimator of the estimated standardized residuals and the null hypothesis of adequacy using a norm. With the L_2 norm and the truncated uniform kernel, we retrieve generalized versions of the classical Box-Pierce/Ljung-Box test statistics. However, using non-uniform kernels, we obtain more powerful test procedures in many situations. The proposed test statistics possess a convenient asymptotic normal distribution under the null hypothesis. We present a simulation experiment and an application

on IBM transaction data.

The third essay investigates the use of tick-by-tick data for market risk measurement. We propose an Intraday Value at Risk (IVaR) at different horizons based on irregularly time-spaced high-frequency data by using an intraday Monte Carlo simulation. An UHF-GARCH model extending the framework of Engle (2000) is used to specify the joint density of the marked point process of durations and high-frequency returns. We apply our methodology to transaction data for the Royal Bank and the Placer Dome stocks traded on the Toronto Stock Exchange. Results show that our approach constitutes reliable means of measuring intraday risk for traders who are very active on the market. The UHF-GARCH model performs well out-of-sample for almost all the time horizons and the confidence levels considered even when normality is assumed for the distribution of the error term, provided that intraday seasonality has been accounted for prior to the estimation.

Keywords: tick-by-tick data, Autoregressive Conditional Duration model, duration clustering, model adequacy, spectral density, marked point process, Intraday Value at Risk, intraday market risk, UHF-GARCH models, intraday Monte Carlo simulation.

Résumé

Cette thèse est constituée de trois essais qui portent sur la modélisation et l'utilisation des données financières transaction par transaction. La finance à haute fréquence est devenue un champ de recherche très actif au cours de deux dernières décennies. Les recherches empiriques utilisant des données de transaction irrégulièrement espacées trouvent leur origine dans le travail de Engle et Russell (1998) introduisant le modèle de durée conditionnelle autorégressive ACD pour l'analyse du temps entre deux événements qui surviennent sur le marché.

Le premier essai propose une revue de la littérature théorique et empirique concernant les modèles ACD. Nous présentons d'abord les propriétés du modèle ACD de base et de ses extensions, les tests de diagnostic existants et les modèles joints d'une durée et d'une caractéristique. Ensuite, nous considérons les applications empiriques des modèles ACD à plusieurs types d'événements et nous identifions des pistes de recherche future.

Le deuxième essai propose deux classes de statistiques de test pour les effets ACD et une classe de statistiques de test pour l'ajustement des modèles ACD, en utilisant une approche spectrale. Les tests d'effets ACD de la première classe sont obtenus en comparant un estimateur à noyau de la densité spectrale normalisée et la densité spectrale normalisée sous l'hypothèse nulle d'absence d'effets ACD, en utilisant une métrique. La deuxième classe de tests d'effets ACD exploite la nature unilatérale de l'hypothèse alternative. La classe de tests d'ajustement d'un modèle ACD est obtenue en comparant un estimateur à noyau de la densité spectrale des résidus estimés standardisés et l'hypothèse nulle d'ajustement en utilisant une métrique. En utilisant le noyau uniforme tronqué et la métrique L_2, nous obtenons des versions généralisées des tests Box-Pierce/Ljung-Box. Cependant, plusieurs noyaux permettent d'obtenir une meilleure puissance. Les statistiques de test proposées possèdent une distribution asymptotique normale rigoureusement établie sous l'hypothèse nulle. Nous réalisons une étude par simulation ainsi qu'une application avec des données de transaction sur l'action IBM.

Dans le troisième essai nous étudions l'utilisation des données transaction par transaction pour mesurer le risque de marché. Nous proposons une Valeur à Risque intrajournalière à horizons différents, basée sur des données à haute fréquence irrégulièrement espacées dans le temps. Les résultats sont obtenus en utilisant une simulation Monte Carlo intrajournalière. La densité jointe du processus de points marqués des durées et des rendements à haute fréquence est spécifiée au moyen d'une extension du modèle UHF-GARCH de Engle (2000). Nous appliquons notre méthodologie à des données sur les actions de la Banque Royale et de Placer Dome transigées à la Bourse de Toronto. Les résultats montrent que notre approche propose une mesure robuste du risque intrajournalier auquel sont confrontés les cambistes très actifs sur le marché. Le modèle UHF-GARCH performe bien hors échantillon pour presque tous les horizons temporels et les degrés de confiance considérés, même lorsque l'hypothèse de normalité est supposée pour la distribution du terme d'erreur, à condition que la saisonnalité intrajournalière ait été prise en compte avant l'estimation.

Mots clés : données transaction par transaction, modèle ACD, effets ACD, ajustement d'un modèle, densité spectrale, processus de points marqués, Valeur à Risque intrajournalière, risque de marché intrajournalier, modèles UHF-GARCH, simulation Monte Carlo intrajournalière

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To my family whose love, sacrifice, and patience have made this project possible,
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Chapter 1

Autoregressive Conditional Duration (ACD) models in finance: A survey of the theoretical and empirical literature

1.1 Introduction

Until two decades ago, most empirical studies in finance employed, as the finest frequency, daily data obtained by retaining either the first or the last observation of the day for the variable of interest (i.e., the closing price), thus neglecting all intraday events. However, due to the increased automatization of financial markets and the rapid developments in raising computer power, more and more exchanges have set up intraday databases that record every single transaction together with its characteristics (such as price, volume etc.). The availability of these low-cost intraday datasets fueled the development of a new area of financial research: high-frequency finance. Embracing finance, econometrics, and time series statistics, the analysis of high-frequency data (HFD) rapidly appeared as a promising avenue for research by facilitating a deeper understanding of market activity.[1] Interestingly, these developments have not been limited to academia, but have also affected the current trading environment. Over the last several years the speed of trading has been constantly increasing. Day-trading, once the exclusive territory of floor-traders, is now available to all investors. "High frequency finance hedge funds" have emerged as a new and successful category of hedge funds.

The intrinsic limit of high-frequency data is represented by the transaction

[1] Econometrics and finance journals (see, for instance, *Journal of Empirical Finance*, 1997; *Journal of Business and Economic Statistics*, 2000; *Empirical Economics*, 2006) have devoted special issues to the examination of high-frequency data; international conferences have focused on this field as well.

or tick-by-tick data in which events are recorded one by one as they arise.[2] Consequently, the distinctive feature of this data is that observations are irregularly time-spaced. This feature challenges researchers as standard econometric techniques, as refined over the years, are no longer directly applicable.[3] Moreover, recent models from the market microstructure literature argue that time may convey information and should, therefore, be modeled as well. Motivated by these considerations, Engle and Russell (1998) developed the Autoregressive Conditional Duration (ACD) model whose explicit objective is the modeling of times between events. Since its introduction, the ACD model and its various extensions have become a leading tool in modeling the behavior of irregularly time-spaced financial data, opening the door to both theoretical and empirical developments. As illustrated by Engle and Russell (1998), Engle (2000), and Engle (2002), the ACD model shares many features with the GARCH model. A decade after its introduction, will it have the same success the GARCH model had in theoretical and empirical studies?

The objective of this paper is to review both the theoretical and empirical work that has been done on ACD models since their introduction a decade ago. ACD models have been partly covered in books such as Bauwens and Giot (2001), Engle and Russell (2002), Tsay (2002), and Hausch (2004), but none of them provides an exhaustive and up-to-date review of the published work on this subject. To our knowledge, this is the first survey-article on ACD models and it aims to offer a good understanding of the scope of current theoretical and empirical research, including recent findings.

The remainder of the paper is organized as follows. Section 1.2 is devoted to the theoretical developments on ACD models. We discuss the properties of the standard ACD specification and several of its extensions, existing diagnostic tests, and joint models for the arrival times of events and some market characteristics.

[2] Engle (2000) denotes them as "ultra-high frequency data".

[3] Other problems are associated with the use of transaction data, such as the bid-ask bounce, the discreteness of prices (see Gwilym and Sutcliffe, 2001, for a review).

Section 1.3 describes the applications of ACD methodology to different types of financial data, depending on the event of interest. Section 1.4 concludes the article.

1.2 The ACD model: theoretical developments

As we have already discussed, the main characteristic of HFD is the fact that they are irregularly time-spaced. Therefore, they are statistically viewed as point processes. A point process is "a special kind of stochastic process, one which generates a random collection of points on the time axis" (Bauwens and Giot, 2001, p.67)[4]. A high-frequency financial dataset contains a collection of financial events such as trades, quotes, etc. and, consequently, the times of these events represent the arrival times of the point process. When different characteristics are associated with an event (such as, for example, the price and the volume associated with a trade), they are called marks, and the double sequence of arrival times and marks is called a marked point process. Point processes are widely used in fields such as queueing theory and neuroscience but have attracted great interest in high-frequency finance over the last few years after Engle (2000) used them as a framework for the analysis of the trading process and of market behavior.

1.2.1 General setup

Let $\{t_0, t_1, ..., t_n, ...\}$ be a sequence of arrival times with $0 = t_0 \leq t_1 \leq ... \leq t_n \leq ...$. Thus, in this general setting simultaneous events are possible but, as discussed in Section 1.3, most of the papers analyzed exclude them. Let $N(t)$ be the number of events that have occurred by time $t \in [0, T]$ and $\{z_0, z_1, ...z_n, ..\}$ the sequence of marks associated with the arrival times $\{t_0, t_1, ..., t_n, ...\}$. Then, $t_{N(T)} = T$ is the last observed point of the sequence and $0 = t_0 \leq t_1 \leq ... \leq t_{N(T)} = T$ corresponds to the observed point process.

[4] A popular point process is the Poisson process. See Hautsch (2004) for a useful review of the different possibilities to classify point processes models.

One common way of studying financial point processes (i.e., point processes that consist of arrival times of events linked to the trading process) is by modeling the process of durations between consecutive points.[5] Let $x_i = t_i - t_{i-1}$ denote the i^{th} duration between two events that occur at times t_{i-1} and t_i. The sequence $\{x_1, x_2, ..., x_{N(T)}\}$ has non-negative elements and this impacts the choice of the appropriate econometric models for durations.

In a very general setup and following Engle's framework (2000), we shall refer to the joint sequence of durations and marks:

$$\{(x_i, z_i), i = 1, ..., T\}.$$

Denote the information set available at time t_{i-1} by $\mathcal{F}_{i-1}$. It includes past durations up to and including x_{i-1} but, as discussed in Section 1.3 it may also contain some pre-determined variables suggested by the microstructure literature. The i^{th} observation has joint density, conditional on $\mathcal{F}_{i-1}$, given by

$$(x_i, z_i)\,|\mathcal{F}_{i-1} \sim f\left(x_i, z_i|\breve{x}_{i-1}, \breve{z}_{i-1}; \theta_f\right), \tag{1.1}$$

where $\breve{x}_{i-1}$ and $\breve{z}_{i-1}$ denote the past of the variables X and Z, respectively, up to the $(i-1)^{th}$ transaction and $\theta_f \in \Theta$ is the set of parameters.

The joint density in (1.1) can be written as the product of the marginal density of the durations and the conditional density of the marks given the durations, all conditioned upon the past of durations and marks:

$$f\left(x_i, z_i|\breve{x}_{i-1}, \breve{z}_{i-1}; \theta_f\right) = g(x_i|\breve{x}_{i-1}, \breve{z}_{i-1}; \theta_x)q(z_i|x_i, \breve{x}_{i-1}, \breve{z}_{i-1}; \theta_z), \tag{1.2}$$

where $g(x_i|\breve{x}_{i-1}, \breve{z}_{i-1}; \theta_x)$ is the marginal density of the duration x_i with parameter

[5] Other ways consist of modeling the intensity process or the counting process, but they are beyond the scope of our paper. For a thorough review of recent intensity models for financial point processes, we refer the interested reader to the book by Hautsch (2004). Winkelmann (1997) and Cameron and Trivedi (1998) provide comprehensive surveys of statistical and econometric techniques for the analysis of count data.

θ_x, conditional on past durations and marks, and $q(z_i|x_i, \breve{x}_{i-1}, \breve{z}_{i-1}; \theta_z)$ is the conditional density of the mark z_i with parameter θ_z, and conditional on past durations and marks as well as the contemporaneous duration x_i. The log-likelihood is given by

$$\mathcal{L}(\theta_x, \theta_z) = \sum_{i=1}^{n} [\log g(x_i|\widetilde{x}_{i-1}, \widetilde{z}_{i-1}; \theta_x) + \log q(z_i|x_i, \widetilde{x}_{i-1}, \widetilde{z}_{i-1}; \theta_z)] \tag{1.3}$$

If the durations are considered weakly exogenous, cf. Engle, Hendry, and Richard (1983), with respect to the processes for the marks, then the two parts of the likelihood function could be maximized separately which simplifies the estimation (see, for instance, Engle, 2000).[6]

1.2.2 Models for the durations

The ACD model introduced by Engle and Russell (1998) can be conceived as a marginal model of durations x_i . Let the conditional expected duration be

$$\psi_i \equiv E(x_i|\mathcal{F}_{i-1}) = \psi_i(\breve{x}_{i-1}, \breve{z}_{i-1}) \tag{1.4}$$

The main assumption of the ACD model is that the standardized durations

$$\varepsilon_i = \frac{x_i}{\psi_i} \tag{1.5}$$

are independent and identically distributed, that is iid with $E(\varepsilon_i) = 1$.[7]

This implies that $g(x_i|\breve{x}_{i-1}, \breve{z}_{i-1}; \theta_x) = g(x_i|\psi_i; \theta_x)$. Thus, all the temporal dependence of the duration process is captured by the conditional expected duration.

Let $p(\varepsilon, \theta_\varepsilon)$ be the density function for ε with parameters θ_ε. The

[6] Whereas no tests for weak exogeneity in this context exist, Dolado, Rodriguez-Poo, and Veredas (2004) recently derived a LM test-statistic useful to apply before separately estimating each density of the joint process.

[7] This assumption is without loss of generality. If $E(\varepsilon_i) \neq 1$, one can define $\varepsilon_i' = \varepsilon_i / E(\varepsilon_i)$ and $\psi_i' = \psi_i E(\varepsilon_i)$ so that $E(\varepsilon_i') = 1$ and $x_i = \psi_i' \varepsilon_i'$.

multiplicative error structure of the model, together with the non-negativity of the duration sequence, requires that $p(\varepsilon, \theta_\varepsilon)$ has a non-negative support.[8] Then, $g(x_i|\mathcal{F}_{i-1};\theta) = \psi_i^{-1} p(x_i/\psi_i;\theta_\varepsilon)$, where $\theta = (\theta_x, \theta_\varepsilon)$ is the vector of all the unknown parameters. The log-likelihood function is given by

$$L(\theta) = \sum_{i=1}^{N(T)} \log g(x_i|\mathcal{F}_{i-1};\theta) = \sum_{i=1}^{N(T)} \left[\log p(\frac{x_i}{\psi_i};\theta_\varepsilon) - \log \psi_i \right]. \tag{1.6}$$

Once a parametric distribution of ε has been specified, maximum likelihood estimates of θ can be obtained by using different numerical optimization algorithms.

The setup in equations (1.4)-(1.5) is very general and allows for a variety of models obtained by choosing different specifications for the expected duration, ψ_i and different distributions for ε. In the following subsections, we shall review the main types of ACD models that have been suggested in the financial econometrics literature as more flexible alternatives to the standard form originally proposed by Engle and Russell (1998).

1.2.2.1 The standard ACD model The basic ACD model as proposed by Engle and Russell (1998) relies on a linear parameterization of (1.4) in which ψ_i depends on m past durations and q past expected durations:

$$\psi_i = \omega + \sum_{j=1}^{m} \alpha_j x_{i-j} + \sum_{j=1}^{q} \beta_j \psi_{i-j}. \tag{1.7}$$

This is referred to as the ACD(m, q) model. To ensure positive conditional durations for all possible realizations, sufficient but not necessary conditions are that $\omega > 0$, $\alpha \geq 0$, $\beta \geq 0$.

As it becomes apparent, the ACD model and the GARCH model of Bollerslev

[8] Another way of dealing with the non-negativity of durations is to specify a (ARMA-type) model for log durations. However, this could pose some problems if some durations are exactly zero.

(1986) share several common features, the ACD model being commonly viewed as the counterpart of the GARCH model for duration data. Both models rely on a similar economic motivation following from the clustering of news and financial events in the markets. The autoregressive form of (1.7) allows for capturing the duration clustering observed in high-frequency data, i.e., small (large) durations being followed by other small (large) durations in a way similar to the GARCH model accounting for the volatility clustering. Just as a GARCH(1,1) is often found to suffice for removing the dependence in squared returns, a low order ACD model is often successful in removing the temporal dependence in durations.

The statistical properties of the ACD(1,1) model are well investigated: Engle and Russell (1998) derive its first two moments while Bauwens and Giot (2000) compute its autocorrelation function.

A very useful feature of the ACD model is that it can be formulated as an ARMA$(\max(m,q),q)$ model for durations x_i. Letting $\eta_i \equiv x_i - \psi_i$, which is a martingale difference by construction and rearranging terms, (1.7) becomes

$$x_i = \omega + \sum_{j=1}^{\max(m,q)} \left(\alpha_j + \beta_j\right) x_{i-j} - \sum_{j=1}^{q} \beta_j \eta_{i-j} + \eta_i. \tag{1.8}$$

It also follows from this ARMA representation that to ensure a well-defined process for durations, all the coefficients in the infinite-order AR representation implied by inverting the MA component must be non-negative; see Nelson and Cao (1992) for derivation of identical conditions to ensure non-negativity of GARCH models. From (1.8), in order for x_i to be covariance-stationary, sufficient conditions are that

$$\sum_{j=1}^{m} \alpha_j + \sum_{j=1}^{q} \beta_j < 1. \tag{1.9}$$

The stationarity and invertibility conditions require that the roots of $[1-\alpha(L)-\beta(L)]$ and $[1-\beta(L)]$, respectively, lie outside the unit circle where $\alpha(L)$ and $\beta(L)$ are the polynomials in terms of the lag operator L. Equation

(1.8) can be used for computing forecasts of durations.

The conditional mean of x_i is by definition (1.4) equal to ψ_i. The unconditional mean of x_i is given by

$$E(x_i) = \frac{\omega}{\left(1 - \sum_{j=1}^{m} \alpha_j - \sum_{j=1}^{q} \beta_j\right)}. \tag{1.10}$$

The conditional variance of x_i based on (1.5) is

$$Var(x_i|\mathcal{F}_{i-1}) = \psi_i^2 Var(\varepsilon_i). \tag{1.11}$$

Thus, the model allows both for conditional overdispersion (when $Var(\varepsilon_i) > 1$) and underdispersion (when $Var(\varepsilon_i) < 1$).[9] Carrasco and Chen (2002) establish sufficient conditions to ensure β-mixing and finite higher-order moments for the ACD(m,q) model. Fernandes (2004) derives lower and upper bounds for the probability density function of stationary ACD(m,q) models.

1.2.2.2 Distributional assumptions and estimation Any distribution defined on a positive support can be specified for $p(\varepsilon, \theta_\varepsilon)$; see Lancaster (1997) for several alternatives. A natural choice very convenient for estimation is the exponential distribution. Engle and Russell (1998) use the standard exponential distribution (that is, the shape parameter is equal to one) which leads to the so-called EACD model. A main advantage of this distribution is that it provides quasi-maximum likelihood (QML) estimators for the ACD parameters (Engle and Russell, 1998; Engle, 2002). Drost and Werker (2004) show that consistent estimates are obtained when the QML estimation is based on the standard gamma family (hence including the exponential).[10] The quasi-likelihood function takes

[9] Dispersion is defined as the ratio of standard deviation to the mean.

[10] Gouriéroux et al. (1984) originally proved that if the conditional mean is correctly specified, even if the density is misspecified, consistent QML estimates can be obtained if and only if the assumed density belongs to the linear exponential family.

the form:

$$L(\theta) = -\sum_{i=1}^{N(T)} \left[\frac{x_i}{\psi_i} + \log \psi_i \right]. \tag{1.12}$$

Following the similarity between the ACD and the GARCH model, the results of Lee and Hansen (1994) and Lumsdaine (1996) on the QMLE properties for the GARCH(1,1) model are formalized for the EACD(1,1) model by Engle and Russell (1998, Corollary, p.1135). Under the conditions of their theorem, consistent and asymptotically normal estimates of θ are obtained by maximizing the quasi-likelihood function given in (1.12), even if the distribution of ε, $p(\varepsilon, \theta_\varepsilon)$, is not exponential.[11] The standard errors need to be adjusted as in Bollerslev and Wooldridge (1992). The corollary also establishes that QML estimates of the ACD parameters can be obtained using standard GARCH software, in particular by considering the dependent variable equal to $\sqrt{x_i}$ and imposing a conditional mean equal to zero. However, a crucial assumption for obtaining QML consistent estimates of the ACD model is that the conditional expectation of durations, ψ_i is correctly specified. We shall discuss this moment restriction later when several types of specification tests for the ACD model are considered. Note also that the corollary is derived for the linear EACD(1,1) model and cannot necessarily be directly extended to more general ACD(m, q) models.

The QML estimation yields consistent estimates and the inference procedures in this case are straightforward to implement, but this comes at the cost of efficiency. In practice, fully efficient ML estimates might be preferred.[12] On the other hand, the choice of the distribution of the error term in (1.5) impacts the conditional intensity or hazard function of the ACD model.[13] The exponential specification implies a flat conditional hazard function which is quite restrictive

[11] The results are also valid for ACD models with unit roots, e.g., integrated models.

[12] This is similar to the ARCH literature where the normal distribution is often rejected in favor of some leptokurtic distribution for returns.

[13] Point processes are frequently formulated in terms of the intensity function. The hazard function is an alternative formulation of the same concept used for cross-sectional data (see Lancaster, 1997 for more information). In the ACD literature the two expressions are used interchangeably.

and easily rejected in empirical financial applications (see Engle and Russell, 1998; Dufour and Engle, 2000a; Feng, Jiang, and Song, 2004; Lin and Tamvakis, 2004, among others). For greater flexibility, Engle and Russell (1998) use the standardized Weibull distribution with shape parameter equal to γ and scale parameter equal to one, the resulting model being called WACD. The Weibull distribution reduces to the exponential distribution if γ equals 1, but it allows for a increasing (decreasing) hazard function if $\gamma > 1$ ($\gamma < 1$). However, Engle and Russell (1998) find evidence of excess dispersion for the Weibull specification. Motivated by a descriptive analysis of empirical volume and price durations, Grammig and Maurer (2000) question the assumption of monotonicity of the hazard function in Engle and Russell's standard ACD models. They advocate the use of a Burr distribution[14] that contains the exponential, Weibull and log-logistic as special cases. The model is then called the Burr-ACD model. It is noteworthy, however, that not all the moments necessarily exist for the Burr distribution unless some restrictions are imposed on the parameters. This, in turn, may sometimes result in poor modeling of the higher (unconditional) moments of durations (see Bauwens, Galli, and Giot, 2003), thus jeopardizing, for example, its use in moment-matching based simulations.[15] Lunde (1999) proposes the use of the generalized gamma distribution which leads to the GACD model. Both the Burr and the generalized gamma distributions allow for hump-shaped hazard functions (they both depend on two parameters) to describe situations where, for small durations, the hazard function is increasing and, for long durations, the hazard function is decreasing. The exact form of the distribution of ε has great importance in some applications of the ACD model, such as the ACD-GARCH class of models in which expected durations enter the conditional heteroskedasticity equation as explanatory variables. Monte Carlo evidence presented by Grammig and Maurer (2000) shows that imposing monotonic

[14] The Burr distribution can be derived as a Gamma mixture of Weibull distributions; see Lancaster (1997).

[15] We refer the reader to Hautsch (2004) for more details on the properties of different distributions used in the ACD framework.

conditional hazard functions when the true data generating process requires non-monotonic hazard functions can have severe consequences for predicting expected durations because the estimators of the parameters of the autoregressive equation tend to be biased and inefficient. Hautsch (2002) specifies different ACD models based on the Generalized F distribution that includes as special cases the generalized gamma, Weibull and log-logistic distributions. Starting from a more general distribution allows one to test if the data support reductions to simpler distributions. It is common to estimate nonparametrically the unconditional distribution of durations x_i and use the shape of this distribution as an indication for choosing the density of ε.[16] More recently, De Luca and Gallo (2004) consider the use of a mixture of two distributions (in particular exponential) justified by the differences in information/behavior among different agents in the market.[17]

As an alternative to specifying a parametric distribution of ε, Engle (2000) and Engle and Russell (1998) use a semiparametric density estimation technique similar to the one used by Engle and Gonzalez-Rivera (1991) in the ARCH context. The conditional mean function of durations is parametrically specified and consistently estimated by QML, and then the baseline hazard is estimated nonparametrically using the standardized durations $\hat{\varepsilon}$ and a k-nearest neighbor estimator.

While crucial for the ACD model and its numerous extensions, the assumption of iid innovations ε_i in (1.5) may be too strong and inappropriate for describing the behavior of some financial durations. As discussed later, Zhang, Russell, and Tsay (2001) relax the independence hypothesis via a regime-switching model. Drost and Werker (2004) drop the iid assumption and consider that innovations ε_i may have dependencies of unknown conditional form, which brings in the question of efficiency in semiparametric estimation. Several semiparametric

[16] In practice, a Gamma kernel approach as proposed by Chen (2000), is frequently used to avoid the boundary bias of fixed kernels; see Grammig and Maurer (2000).

[17] The mixing parameter then has a financial interpretation as the proportion of informed (uninformed) agents but restrictively assumes that such a proportion is constant in a given time interval; see De Luca and Gallo (2004) for further discussion.

alternatives may be considered, such as Markov innovations and martingale innovations. The iid case of innovations with unknown density is also obtained under suitable restrictions. Using the concept of the efficient score function from the literature on semiparametric estimation,[18] the authors illustrate that even small dependencies in the innovations can trigger considerable efficiency gains in the efficient semiparametric procedures over the QML procedure.

The standard ACD model has been extended in several ways, directed mainly to improving the fitting of the stylized facts of financial durations. The strong similarity between the ACD and GARCH models nurtured the rapid expansion of alternative specifications of conditional durations. In the next subsection we review the most popular generalizations that have been proposed in the literature as well as some of the more recent and promising models.

1.2.2.3 Extensions of the standard ACD model

Persistence in durations. As we shall discuss in Section 1.3, empirical studies based on the linear model in (1.7) often reveal persistence in durations as the estimated coefficients on lagged variables add up nearly to one. Moreover, many financial duration series show a hyperbolic decay, i.e., significant autocorrelations up to long lags. This suggests that a better fit might be obtained by accounting for longer term dependence in durations. Indeed, the specification given by (1.8) shows that the standard ACD model imposes an exponential decay pattern on the autocorrelation function typical for stationary and invertible ARMA processes. Or, this may be completely inappropriate in the presence of long memory processes. While the long-memory phenomenon has been extensively studied both in the theoretical and in the applied literature for time series and volatility[19], it has received considerably less attention in the literature on ACD models. Jasiak (1998) proposes the Fractionally Integrated

[18] See, for instance, Drost et al. (1997).

[19] See Baillie (1996) for a review, or Banerjee and Urga (2005) for a survey of more recent developments in the analysis of long-memory.

ACD (FIACD) model, analogous to the FIGARCH model of Baillie, Bollerslev, and Mikkelsen (1996). The FIACD(m, d, q) model is defined as:

$$[1-\beta(L)]\,\psi_i = \omega + \left[1-\beta(L)-[1-\phi(L)]\,(1-L)^d\right] x_{i,} \tag{1.13}$$

where $\phi(L) = \alpha(L) + \beta(L)$ and the fractional differencing operator $(1-L)^d$ is given by $(1-L)^d = \sum_{k=0}^{\infty} \Gamma(k-d)\Gamma(k+1)^{-1}\Gamma(-d)^{-1}L^k$, Γ denoting the gamma function and $0 < d < 1$. When $d = 1$, the model is called Integrated ACD (IACD) by analogy with the IGARCH model of Engle and Bollerslev (1986). Building on the stationarity and ergodicity conditions derived by Bougerol and Picard (1992) for the IGARCH model, it can be shown that the FIACD model is strictly stationary and ergodic (see Jasiak, 1998 for further details).

As is well known from the literature on time series, long-memory may also occur because of the presence of structural breaks or regime-switching in the series. We shall later discuss some of the regime-switching models that have been proposed in the ACD literature.

Logarithmic-ACD models and asymmetric news impact curves. In the ACD(m, q) model (1.7) sufficient conditions are required for the parameters to ensure the positivity of durations. If one wants to add linearly in the autoregressive equation some variables taken from the microstructure literature and having expected negative coefficients, the durations might become negative. To avoid this situation, Bauwens and Giot (2000) introduce the more flexible logarithmic-ACD or Log-ACD(m, q) model in which the autoregressive equation is specified on the logarithm of the conditional duration ψ_i. The model is defined by (1.5) and one of the two proposed parameterizations of (1.4), referred as Log-

ACD_1 and Log-ACD_2, respectively[20]:

$$\ln \psi_i = \omega + \sum_{j=1}^{m} \alpha_j \ln x_{i-j} + \sum_{j=1}^{q} \beta_j \ln \psi_{i-j} = \omega + \sum_{j=1}^{m} \alpha_j \ln \varepsilon_{i-j} + \sum_{j=1}^{q} (\beta_j - \alpha_j) \ln \psi_{i-j}, \tag{1.14}$$

$$\ln \psi_i = \omega + \sum_{j=1}^{m} \alpha_j \varepsilon_{i-j} + \sum_{j=1}^{q} \beta_j \ln \psi_{i-j} = \omega + \sum_{j=1}^{m} \alpha_j (x_{i-j}/\psi_{i-j}) + \sum_{j=1}^{q} \beta_j \ln \psi_{i-j}. \tag{1.15}$$

Unlike the standard ACD(m, q) model, no non-negativity restrictions on the parameters of the autoregressive equation are needed to ensure the positivity of conditional durations. Covariance stationarity conditions are necessary ($|\alpha + \beta| < 1$ for the Log-ACD_1 model and $|\beta| < 1$ for the Log-ACD_2 model). In practice, the Log-ACD_2 model is often preferred as it seems to fit the data better. Bauwens, Galli, and Giot (2003) derive analytical expressions for the unconditional moments and the autocorrelation function of Log-ACD models. Unlike the ACD model whose autocorrelation function decreases geometrically at the rate $\alpha + \beta$ (see equation 1.8), the autocorrelation function of the Log-ACD model decreases at a rate less than β for small lags (see Bauwens, Galli, and Giot, 2003). The same probability distribution functions can be chosen for ε as in the ACD model, and the estimation can be analogously carried by maximum likelihood, considering the new definition of the conditional duration.

Tests for nonlinearity conducted by Engle and Russell (1998) on IBM trade duration data suggest that the standard ACD model given by (1.5) and (1.7) cannot fully capture nonlinear dependence between conditional duration and past information set. In particular, the authors report that conditional durations are overpredicted by the linear specification after very short or very long durations. This opened the door for new models looking for more flexible functional forms

[20] Readers familiar with the GARCH literature will notice that the Log-ACD_1 model is analogous to the Log-GARCH model of Geweke (1986), while the Log-ACD_2 model resembles the EGARCH model proposed by Nelson (1991).

that allow for distinctive responses to small and large shocks. As can be easily seen in (1.14), the Log-ACD$_1$ model also allows for nonlinear effects of short and long durations (i.e. when $\varepsilon_i < 1$ or $\varepsilon_i > 1$) without including any additional parameters. Dufour and Engle (2000a) argue that the Log-ACD model is likely to produce an over-adjustment of the conditional mean after very short durations because of the asymptotic convergence to minus infinity of log(0). Instead, they propose to model the news impact function with a piece-wise linear specification. The model is called the EXponential ACD or EXACD model due to its similarity to the EGARCH specification proposed by Nelson (1991), and it specifies the conditional duration as an asymmetric function of past durations:

$$\ln \psi_i = \omega + \sum_{j=1}^{m} \left[\alpha_j \varepsilon_{i-j} + \delta_j \left|\varepsilon_{i-j} - 1\right|\right] + \sum_{j=1}^{q} \beta_j \ln \psi_{i-j}. \tag{1.16}$$

Thus, the impact on the conditional duration is different, depending on the durations being shorter or longer than the conditional mean, i.e., $\varepsilon_i < 1$ involves a slope equal to $\alpha - \delta$ while $\varepsilon_i > 1$ determines a slope equal to $\alpha + \delta$.

Following the approach taken by Hentschel (1995) to developing a class of asymmetric GARCH models, Fernandes and Grammig (2006) introduce an interesting class of augmented ACD (AACD) models that encompass most of the specifications mentioned before as well as some models that have not been considered yet.[21] The AACD models are obtained by applying a Box-Cox transformation to the conditional duration process and a nonlinear function of ε_i that allows asymmetric responses to small and large shocks. The lowest-order parametrization is given by:

$$\psi_i^{\lambda} = \omega + \alpha \psi_{i-1}^{\lambda} \left[\left|\varepsilon_{i-1} - b\right| + c\left(\varepsilon_{i-1} - b\right)\right]^{v} + \beta \psi_{i-1}^{\lambda}. \tag{1.17}$$

In this specification, the asymmetric response to shocks and the shape of the piecewise function depend on parameters b and c, respectively, while parameter v

[21] Note, however, that they do not nest the TACD and STACD models presented later.

induces concavity (convexity) of the shock impact curve for $\upsilon \leqslant 1$ ($\upsilon \geqslant 1$).

Building on the theoretical developments of Carrasco and Chen (2002), Fernandes and Grammig (2006) derive sufficient conditions ensuring the existence of higher-order moments, strict stationarity, geometric ergodicity and β-mixing.[22] Under suitable restrictions several ACD models can be recovered: the standard ACD model, the Log-ACD models, the EXACD model, as well as other specifications inspired by the GARCH literature. Depending on the choice of the distribution of ε_i, the models can be estimated by maximum likelihood.

Regime-switching ACD models. Despite the evidence of nonlinearity reported by several studies (see Dufour and Engle, 2000a; Zhang, Russell, and Tsay, 2001; Taylor, 2004; Fernandes and Grammig, 2006; Meitz and Teräsvirta, 2006) the question of the type of nonlinear ACD model that would be the most appropriate needs further investigation. An alternative approach to dealing with the nonlinearity aspect-behavior evidenced by Engle and Russell (1998) involves considering the existence of different regimes with different dynamics corresponding to heavier or thinner trading periods. For instance, Zhang, Russell, and Tsay (2001) introduce a threshold ACD or TACD($m.q$) model in which the conditional duration depends nonlinearly on past information variables.[23] A K-regime TACD(m,q) model is given by

$$\left\{ \begin{array}{c} x_i = \psi_i \varepsilon_i^{(k)} \\ \psi_i = \omega^{(k)} + \sum\limits_{j=1}^{m} \alpha_j^{(k)} x_{i-j} + \sum\limits_{j=1}^{q} \beta_j^{(k)} \psi_{i-j} \end{array} \right. \qquad \text{if } x_{i-1} \in R_k, \tag{1.18}$$

where $R_k = [r_{k-1}, r_k]$, $k = 1, 2, ..., K$, with $K \in \mathbb{Z}^+$ being the number of regimes and $0 = r_0 < r_1 < ... < r_K = \infty$ are the threshold values. The regime-switching ACD parameters are denoted by $\omega^{(k)} > 0$, $\alpha_j^{(k)} \geqslant 0$ and $\beta_j^{(k)} \geqslant 0$. Moreover,

[22] Conditions for β-mixing and existence of moments for nonlinear ACD structures have been investigated by Meitz and Saikkonen (2004) for the first order case.

[23] The TACD model can be viewed as a generalization of the threshold-GARCH model of Rabemananjara and Zakoian (1993) and Zakoian (1994).

for a fixed k, the error term $\varepsilon_i^{(k)}$ is an iid sequence with positive distribution that is regime-specific[24]. Consequently, the different regimes of a TACD model have different duration persistence, conditional means and error distributions, which allows for greater flexibility compared to the ACD model. Conditions for geometric ergodicity and existence of moments are derived only for the TACD(1,1) model but they are difficult to generalize for higher order models. Moreover, the impact of the choice of the threshold variable still needs to be assessed. For a large number of regimes K, the estimation of the model may become computationally intensive[25] since it is performed by using a grid search across the threshold values r_1 and r_2 and by maximizing the likelihood function for each pair. An interesting finding of the Zhang, Russell, and Tsay (2001) study concerns the identification in the data analyzed of multiple structural breaks corresponding to some economic events. Following location of these break points, separate models are to be estimated for each subperiod.

The question of nonstationarity is addressed differently by Meitz and Teräsvirta (2006) who propose using ACD models with parameters changing smoothly over time. The logistic function is used as the transition function with the time as transition variable. Depending on the definition of time considered, time-varying ACD or TVACD models can be usefully applied either to test against parameter changes or to detect unsatisfactory removal of the intraday seasonality. The same authors also introduce an alternative to the TACD model. Building on the literature on smooth transition GARCH models,[26] they advocate the use of a smooth transition version of the ACD model, the STACD model, in which the transition between states is driven by a transition function. A logarithmic version may also be specified.

[24] The authors use the generalized gamma distribution. Note also that the choice of the threshold variable is not restricted to lag-1 duration.

[25] Zhang, Russell, and Tsay (2001) estimate a 3-regime TACD(1,1) model while Bauwens, Giot, Grammig, and Veredas (2004) employ a logarithmic version of the same model.

[26] See, for example, Hagerud (1996), Gonzalez-Rivera (1998) or Lundbergh and Teräsvirta (2002).

Latent factor-based models. Over recent years an area garnering substantial interest in the asset return literature has been that of stochastic volatility (SV) models. Modeling volatility as an unobserved latent variable has shown itself capable of capturing the dynamics of the financial series of returns in a better way than the competing GARCH models.[27] Given the similarity between ACD and GARCH models, it is not surprising that recent research on ACD models has taken this route. Bauwens and Veredas (2004) propose the stochastic conditional duration (SCD) model in which the conditional duration ψ_i is modeled as a latent variable (instead of being deterministic as in the ACD model). Economically, the latent variable may be thought of as capturing the unobservable information flow in the market. The SCD model as it was first proposed is given by

$$x_i = \psi_i \varepsilon_{i,} \tag{1.19}$$

$$\ln \psi_i = \omega + \beta \ln \psi_{i-1} + u_i, \qquad |\beta| < 1. \tag{1.20}$$

While equation (1.19) is identical to equation (1.5), the conditional duration in (1.20) is driven by a stationary first order autoregressive process AR(1).[28] Thus, the model has two sources of uncertainty, that is, ε_i for observed duration and u_i for conditional duration, which is expected to offer greater flexibility in describing the dynamics of the duration process. The following distributional assumptions are made:

$$\varepsilon_i \mid \mathcal{F}_{i-1} \sim iid\ p(\varepsilon_i) \quad \text{and} \quad u_i \mid \mathcal{F}_{i-1} \sim iid\ \mathcal{N}(0, \sigma^2) \quad \text{with } u_i \text{ independent of } \varepsilon_i. \tag{1.21}$$

[27] See, for example, Danielsson (1994); Kim, Shephard, and Chib (1998); Ghysels, Harvey, and Renault (1996) on the advantages of using the SV framework relative to the GARCH framework.

[28] One may notice that no non-negativity constraints are imposed on the parameters to insure the positivity of durations. Also, the model mixes features of both the standard ACD model and the SV model introduced by Taylor (1982).

The same distribution may be chosen for ε_i as in the ACD framework and a different distribution for u_i can also be assumed. The original specification uses the Weibull and gamma distributions for ε_i but similar results are reported for both. Compared to the ACD model, the SCD model can generate a wider variety of hazard functions shapes (see Bauwens and Veredas, 2004 for further details). It follows from the specification in (1.19)-(1.21) that the SCD model is a mixture model. Therefore, its estimation is quite challenging because the exact likelihood function cannot be derived in a closed form but involves a multidimensional integral whose evaluation requires extensive simulations, especially for large datasets (which is usually the case when working with high-frequency data).

Bauwens and Veredas (2004) propose use of the QML method with the Kalman filter after putting the model in a linear state space representation.[29] The procedure provides consistent and asymptotically normal estimators, but it is inefficient as it does not rely on the true likelihood of durations x_i. Strickland, Forbes, and Martin (2005) employ the Monte Carlo Markov Chain (MCMC) methodology which, like the QML is a complete method (i.e., permits estimation of both parameters and latent variables) but its implementation is rather time consuming.[30] An alternative method for the estimation of the SCD model is currently under investigation by Bauwens and Galli (2005) following the work of Liesenfeld and Richard (2003) on the application of the efficient importance sampling procedure to SV models. Ning (2004) investigates two other methods proposed in the SV literature: the empirical characteristic function and the GMM methods.

The leverage effect is well known in the volatility literature since Black (1976) first noted a negative correlation between current returns and future volatility. Feng, Jiang, and Song (2004) further develop the idea of asymmetric behavior of

[29] This procedure has been proposed independently by Nelson (1988) and Harvey, Ruiz, and Shephard (1994) for SV models.

[30] The SV literature shows the MCMC to have been more efficient than the QML and the generalized method of moments (GMM) techniques; see, for instance, Jacquier, Poison, and Rossi (1994).

the expected duration in a SCD framework. Their model is characterized by the introduction of an intertemporal term ($ln\varepsilon_{i-1}$) in the latent function given in (1.20) to account for a leverage effect. The empirical study finds evidence of a positive relation between trade duration and conditional expected duration. The model is estimated using the Monte Carlo maximum-likelihood (MCML) method proposed by Durbin and Koopman (1997) in the general framework of non-gaussian state space models, and applied by Sandmann and Koopman (1998) to the estimation of SV models.

Given the rich potential of specifications and methods of estimation existing in the literature on SV models (see Broto and Ruiz, 2004 for a recent survey), we anticipate further developments on SCD models in the near future. Formal comparisons of alternative estimation methods would be interesting.

As evidenced by (1.4) and (1.11), the ACD model does not allow for independent variation of the conditional mean and variance as higher order conditional moments are linked to specification of the conditional mean. Ghysels, Gouriéroux, and Jasiak (2004), referred to as GGJ hereafter, argue that this is a very restrictive assumption, especially when one is interested in analysis of market liquidity. Intertrade durations are an indicator of market liquidity because they measure the speed of the market but the variance of durations describes the risk on time associated to the liquidity risk. Therefore, GGJ introduce the Stochastic Volatility Duration (SVD) model in which the dynamics of the conditional mean and variance are untied by using two time varying factors instead of one. The SVD model extends the standard static exponential duration model with gamma heterogeneity from the literature on cross-sectional and panel data and given by $x_i = u_i/av_i$, where u_i and v_i are two independent variables with distributions standard exponential and gamma, respectively. This model can then be rewritten in terms of Gaussian factors as

$$x_i = \frac{H(1, F_{1i})}{aH(b, F_{2i})} \tag{1.22}$$

where a and b are positive parameters, F_{1i} and F_{2i} are iid standard normal variables, and $H(b, F) = G(b, \phi(F))$ where $G(b, .)$ is the quantile function of the gamma(b, b) distribution and $\phi()$ the c.d.f. of the standard normal distribution. GGJ propose to introduce dynamic patterns through the two underlying Gaussian factors by considering a VAR representation for the process $F_i = (F_{1i}, F_{2i})'$. The marginal distribution of F_i is constrained to be $N(0, I)-$ I being the identity matrix - to ensure that the marginal distribution of durations x_i belongs to the class of exponential distributions with gamma heterogeneity (i.e. Pareto distributions). Thus, the SVD model is given by (1.22) and

$$F_i = \sum_{j=1}^{p} \Lambda_j F_{i-j} + \varepsilon_i \tag{1.23}$$

where Λ_j is the matrix of autoregressive VAR parameters and ε_i is a Gaussian white noise with variance-covariance matrix $\sum(\Lambda)$ such that $Var(F_i) = I$.

While conceptually interesting, the use of the SVD model in applied research has been limited, due to its rather complicated estimation procedure. Indeed, the likelihood function is difficult to evaluate which is typical of the class of nonlinear dynamic factor models. GGJ suggest a two-step procedure in which parameters a and b are first estimated by QML, exploiting the fact that the marginal distribution of x_i is a Pareto distribution that depends only on the parameters a and b, and then the method of simulated moments[31] is used to get the autoregressive parameters Λ_j (see GGJ, 2004 for more details). It is noteworthy that the assumption of a Pareto distribution for the marginal distribution of x_i may be completely inappropriate for some duration processes, as evidenced by Bauwens, Giot, Grammig, and Veredas (2004) and this makes the first step of the estimation procedure unfeasible. Moreover, the same study finds a poorer predictive performance of the SVD model compared to ACD and Log-ACD models. More formal investigations of these alternative specifications

[31] See, for example, Gouriéroux and Monfort (1997).

for nonlinearity would be interesting.

1.2.2.4 Tests of the ACD model A major issue when using ACD models is how to assess the adequacy of the estimated model. Even though, as discussed before, a variety of ACD specifications have been proposed in the literature, the question of evaluating a particular model has attracted far less interest. Most of the papers surveyed limit the testing to simple examinations of the standardized residuals. Recent papers, however, have proposed useful procedures for testing either the specification of the conditional mean function given in (1.4) or the specification of the distribution of the standardized durations $p(\varepsilon, \theta_\varepsilon)$ in (1.5). In the following, we briefly review these different approaches.

Basic residual examinations. The common way of evaluating ACD models consists of examining the dynamical and distributional properties of the estimated standardized residuals of an ACD model

$$\hat{\varepsilon}_i = \frac{x_i}{\hat{\psi}_i}, \quad i = 1, ..., T.$$

If the estimated model for the durations series is adequate, it follows that $\hat{\varepsilon}_i$ are iid. The approach used by Engle and Russell (1998) and largely adopted by subsequent authors consists of applying the Ljung-Box Q-statistic (see Ljung and Box, 1978) to the estimated residuals $\hat{\varepsilon}_i$ and to the squared estimated residuals $\hat{\varepsilon}_i^2$ to check for remaining serial dependence.[32] Practically all the papers surveyed employed this procedure. However, it is noteworthy that this approach is questionable. While the Ljung-Box test statistic is assumed to have an asymptotic χ^2 distribution under the null hypothesis of adequacy, no formal analysis exists that rigorously establishes this result in the context of ACD models. In fact, in a related context, Li and Mak (1994) show that this test statistic does not have the usual asymptotic

[32] When the Ljung-Box statistic is applied to the squares of the estimated residuals, it is known as the McLeod and Li (1983) test.

χ^2 distribution under the null hypothesis when it is applied to standardized residuals of an estimated GARCH model. In the ACD framework, Li and Yu (2003) follow Li and Mak (1994) and propose a corrected statistic that results in a portmanteau test for the goodness-of-fit when ε_i follows the exponential distribution.[33] Like in the time series literature, additional examinations of residuals include visual check of the autocorrelation function of estimated residuals (see, for instance, Jasiak, 1998; Bauwens and Giot, 2000; Ghysels, Gouriéroux, and Jasiak, 2004; Bauwens, 2006). Furthermore, some papers (for example, Bauwens and Veredas, 2004; Ghysels, Gouriéroux, and Jasiak, 2004) compare the marginal density of durations derived from the model to the empirical marginal density directly obtained from the observed durations.

Beside testing for serial dependence in the estimated residuals, a few papers also test the moments conditions implied by the specified distribution of ε_i. According to the ACD model, the estimated residuals should have a mean of one. Engle and Russell (1998) propose a test for no excess dispersion of the estimated residuals when an exponential or Weibull distribution are assumed.[34] Bauwens and Veredas (2004) and De Luca and Gallo (2004) use QQ-plots to check the distribution assumptions while Prigent, Renault, and Scaillet (2001) employ Bartlett identity tests.

Testing the functional form of the conditional mean duration. As we have already mentioned, the validity of the conditional mean function is essential for the QML estimation of the ACD model. Sophisticated tests of no remaining ACD effects with rigorously proven asymptotic distributions have been recently developed in the literature. Following the work of Lundbergh and Teräsvirta (2002) on GARCH models who propose a Lagrange multiplier test of no residual

[33] According to Li and Yu (2003) the case of a Weibull distribution can be analyzed in the same way via the change of a variable.

[34] In a Monte Carlo exercise, Fernandes and Grammig (2005) find poor performance of this overdispersion test compared to the nonparametric tests they propose.

ARCH,[35] Meitz and Teräsvirta (2006) propose a similar statistic of no residual ACD which is shown to be asymptotically equivalent to the Li and Yu (2003) test and also has a version robust to deviations from the assumed distribution. Alternatively, under the null hypothesis of adequacy of an ACD model, the fact that the estimated residuals $\overset{\curlywedge}{\varepsilon}_i$ are iid implies that their normalized spectral density equals the flat spectrum (that is, $1/2\pi$). Consequently, adequacy test statistics may be constructed by comparing an estimator of the spectral density of the estimated residuals and the flat spectrum. Building on Hong (1996, 1997), Duchesne and Pacurar (2005) make use of a kernel-based estimator of the normalized spectral density of the estimated residuals to construct such adequacy tests. Interestingly, a generalized version of the classic Box-Pierce/Ljung-Box test statistics is obtained as a special case (and therefore possesses a proven asymptotical distribution) but is shown to be less powerful. In a similar way, Duchesne and Hong (2001) use a wavelet-based estimator with a data-driven smoothing parameter.

The model presented in (1.7) assumes a linear dependence of the conditional duration on past information set. Engle and Russell (1998) propose a simple test for detecting potential nonlinear dependencies that is also applied by Zhang, Russell, and Tsay (2001) to motivate the introduction of their TACD model. The idea is to divide the durations into bins ranging from 0 to ∞ and then regress the estimated residuals on indicators of the size of the previous duration. Under the null hypothesis of the estimated residuals being iid, the coefficient of determination of this regression should be zero while under the alternative hypothesis, one can analyze the coefficients of indicators that are significant in order to identify sources of misspecification.

Meitz and Teräsvirta (2006) develop a useful and very general battery of tests of Lagrange multiplier type that allow extensive checks against different forms of misspecification of the functional form of the conditional mean duration:

[35] They also show that this test is asymptotically equivalent to the Li and Mak (1994) test.

tests against higher-order ACD models, tests of linearity, and tests of parameters constancy. Finally, the omnibus procedure suggested by Hong and Lee (2003) for a large class of time series models and based on the generalized spectral density may also be used as a misspecification test of ACD models. As such, it is shown to be consistent against any type of pairwise serial dependence in the standardized durations.

Testing the distribution of the error term. Another possible source of misspecification for ACD models is the distribution of the error term. While the exponential distribution (or other member of the standard gamma family as shown by Drost and Werker, 2004) leads to consistent QML estimates, this procedure may be unsatisfactory in finite samples, as we have already mentioned. Only two of the theoretical papers surveyed paid attention to the development of elaborate tests against distributional misspecification. Fernandes and Grammig (2005) introduce two tests for distribution of the error term (the so-called D-test and H-test) based on comparison between parametric and nonparametric estimates of the density and of the hazard rate function of the estimated residuals, respectively. These tests inspect the whole distribution of the residuals, not only a limited number of moment restrictions, and are shown to be nuisance parameter free. It is noteworthy however, that the conditional mean function is assumed to be correctly specified and, therefore, a rejection could follow also from a misspecified conditional mean.

Another way to test ACD models consists of using the framework developed by Diebold, Gunther, and Tay (1998) for the evaluation of density forecasts. The main idea is that the sequence of probability integral transforms of the one-step-ahead density forecast has a distribution iid uniform $U(0,1)$ under the null hypothesis that the one-step-ahead prediction of the conditional density of durations is the correct density forecast for the data-generating process of durations. It follows that standard tests for uniformity and iid may then be used, but they do not generally indicate potential causes of the rejection of the null hypothesis. Therefore, the aforementioned authors recommend the use of graphical procedures such as examinations of histograms and autocorrelograms. This approach is adopted by Bauwens, Giot, Grammig, and Veredas (2004) for comparing the predictive abilities of the most popular ACD specifications. In the same context, Dufour and Engle (2000a) propose a new Lagrange Multiplier type of test for the uniformity and iid assumptions. Both approaches, however, assume the right conditional mean parameterization and, consequently, possible rejections may be due either to violation of distributional assumptions or violation of the conditional mean restriction.

Summing up, since existing misspecification tests of ACD models are directed toward detecting either distributional misspecification or functional misspecification, attention should be given in practice to the use of several complementary tests so that different sources of problems might be recognized. Unfortunately, none of these sophisticated procedures, to our knowledge has made its way to empirical studies so far.

1.2.3 Models for durations and marks

The models discussed above have been proposed for describing the dynamics of durations, x_i with respect to past information. In this respect, they are marginal models, following from the decomposition given by (3.2). But one may also be interested in the behavior of some marks (e.g., price, volume) given the durations and the past information, especially for testing some predictions from the market microstructure literature, as we shall discuss in the next section. In this case, the structure given by (3.2) provides a suitable framework for the joint modeling of durations between events of interest, x_i and some market characteristics, z_i. In most cases, only one mark is considered: the price.

The first joint model for durations and prices has been developed by Engle

(2000).[36] He introduces an ACD-GARCH model based on the decomposition (3.2), the so-called Ultra-High-Frequency (UHF)-GARCH model. Durations between transactions are described by an ACD-type model conditional on the past while price changes are described by a GARCH model adapted to irregularly time-spaced data conditional on contemporaneous durations and the past. The adaptation consists of measuring the volatility per unit time. Under the assumption of weakly exogenous durations, the ACD model can be estimated first (cf. Engle, Hendry, and Richard; 1983), and then the contemporaneous duration and expected duration enter the GARCH model for volatility together with some other explanatory variables. As discussed later, the model can be used to investigate the relationship between current durations and volatility.

However, insights from the market microstructure literature suggest that it is possible that the volatility also impacts the duration process and ignoring this issue neglects part of the complex relationship between durations and volatility. Grammig and Wellner (2002) extend Engle (2000) approach by formulating a model for the interdependence of intraday volatility and duration between trades: the interdependent duration-volatility (IDV) model.

An alternative ACD-GARCH specification has been proposed by Ghysels and Jasiak (1998) based on the results of Drost and Nijman (1993) and Drost and Werker (1996) on the temporal aggregation of GARCH processes.[37] The model is formulated as a random coefficient GARCH where the parameters are driven by an ACD model for the durations between transactions. A GMM procedure is suggested for estimating the model but its application is rather computationally complex, which could explain the limited interest this model has received in empirical applications.

Whereas the models discussed above assume continuous distributions for price changes, other models have been put forward in the literature to specify the joint

[36] The working paper version dates from 1996.

[37] For a formal comparison of the UHF-GARCH and ACD-GARCH models, together with their advantages, we refer the interested reader to the work of Meddahi, Renault, and Werker (2006).

distribution of durations and price changes represented as a discrete variable. Bauwens and Giot (2003) and Russell and Engle (2005) propose competing risk models or transition models from the previous price change to the next one. The asymmetric Log-ACD model of Bauwens and Giot (2003) makes the durations dependent upon the direction of the previous price change, hence the asymmetric side. A Log-ACD model is used to describe the durations between two bid/ask quotes posted by a market maker but the model takes into account the direction of the mid bid/ask price change between the beginning and the end of a duration through a binary variable. Intraday transaction prices often take just a limited number of different values due to some institutional features with regard to price restrictions.[38] Therefore, Russell and Engle (2005) introduce the Autoregressive Conditional Multinomial (ACM) model to account for the discreteness of transaction prices. Building on (3.2) they propose using an ACD model for durations and a dynamic multinomial model for distribution of price changes conditional on past information and the contemporaneous duration. However, depending on the number of state changes considered, a large number of parameters might be needed, which complicates the estimation. A two-state only ACM model has been subsequently applied by Prigent, Renault, and Scaillet (2001) to option pricing.

Other approaches for studying causality relationships between durations and different marks have been developed based on the vector autoregressive (VAR) system used by Hasbrouck (1991). Hasbrouck (1991) analyzes the price impact of a trade on future prices but without assuming any influence of the timing of transactions on the distribution of marks, that is, $q(z_i|x_i, \breve{x}_{i-1}, \breve{z}_{i-1}; \theta_z) = q(z_i|\breve{z}_{i-1}; \theta_z)$ in (3.2). A simple bivariate model for changes in quotes and trade dynamics (e.g., trade sign) can be specified in transaction time. Dufour and Engle (2000b) extend Hasbrouck's model to allow durations to have an impact on price changes. An ACD model describes the dynamics of durations and

[38]It has been also reported (see, for instance, Tsay, 2002; Bertram, 2004; Dionne, Duchesne, and Pacurar, 2005) that a large percentage of intraday transactions have no price change.

the duration between transactions is then treated as a predetermined variable by allowing coefficients in both price changes and trade equations to be time-varying, depending on the duration. The model can easily be extended to include other variables, such as volume and volatility (Spierdijk, 2004; Manganelli, 2005).

An important assumption typically made in these joint models of durations and marks is that durations have some form of exogeneity (weak or strong, cf. Engle, Hendry, and Richard, 1983) which greatly simplifies the estimation and forecast procedures, respectively (see, for instance, Engle, 2000; Ghysels and Jasiak, 1998; Dufour and Engle, 2000b; Manganelli, 2005). The empirical consequences of removing the exogeneity assumption in VAR-type models are still open to question.[39]

1.3 Applications of ACD models in finance

As already mentioned in Section 1.2, ACD models can be used for modeling of arrival times of a variety of financial events. Most applications focus on the analysis of the trading process based on trade and price durations (as first initiated by Engle and Russell, 1997, 1998, and defined below) but other economic events, such as firm defaults or interventions of the Central Bank have also been considered. However, the importance of ACD models in finance so far stems from the relatively recent market microstructure literature that provides a strong economic motivation for use of these models besides the statistical justification already discussed (i.e., data being irregularly time-spaced). In the following, we present a review of the empirical applications of ACD methodology sorted according to the type of the event of interest. But first, we shall discuss an essential feature of all types of intraday durations.

[39] Moreover, Dufour and Engle (2000b, p. 2493) also state that "we do not have knowledge, to this date, of any theoretical model that shapes the reciprocal interactions of price, trade, and time...".

1.3.1 Intraday seasonality

It is a well known fact that within a trading day financial markets are characterized by a strong seasonality. Initial intraday studies reporting seasonality used data resampled at regular time intervals (e.g., hourly, every half hour, every 5 minutes, etc.) and focused mainly on the behavior of the intraday volatility (see, for instance, Bollerslev and Domowitz, 1993; Andersen and Bollerslev, 1997, 1998; Beltratti and Morana, 1999). In the context of irregularly time-spaced intraday data, Engle and Russell (1998) report higher trading activity (hence, shorter durations on average) at the beginning and close of the trading day, and slower trading activity (longer durations) in the middle of the day. These patterns are linked to exchange features and traders' habits. Traders are very active at the opening as they engage in transactions to benefit from the overnight news. Similarly, at the closing, some traders want to close their positions before the end of the session. Lunchtime is naturally associated with less trading activity. Ignoring these intraday patterns would distort any estimation results. The procedure used most often in this literature for taking into account intraday seasonal effects is that originally employed by Engle and Russell (1998). It consists of decomposing intraday durations into a deterministic part based on the time of the day the duration arises, and a stochastic part modeling the durations' dynamics:

$$x_i = \widetilde{x}_i s\left(t_{i-1}\right), \tag{1.24}$$

where $\widetilde{x}_i$ denotes the "diurnally adjusted" durations and $s\left(t_i\right)$ denotes the seasonal factor at t_i. Equation (1.4) for the conditional duration becomes

$$\psi_i = E(\widetilde{x}_i|\mathcal{F}_{i-1})s\left(t_{i-1}\right). \tag{1.25}$$

The two set of parameters of the conditional mean and of the seasonal factor, respectively, can be jointly estimated by maximum likelihood as in Engle and Russell (1998). However, due to numerical problems arising when trying to achieve convergence, it is more common to apply a two-step procedure in which the raw durations are first diurnally adjusted and then the ACD models are estimated on the deseasonalized durations, $\widetilde{x}_i$. Engle and Russell (1998) argue that both procedures give similar results due to the large size of intraday datasets.

With regard to the specification of the seasonal factor $s(t)$, two similar approaches are dominant in the empirical studies. The first is the one originated by Engle and Russell (1997, 1998) in which durations x_i are regressed on the time of the day using a piecewise linear spline or cubic spline specification and then diurnally adjusted durations $\widetilde{x}_i$ are obtained by taking the ratios of durations to fitted values. Second, Bauwens and Giot (2000) define the seasonal factor as the expectation of duration conditioned on time-of-day. This expectation is computed by averaging the raw durations over thirty-minute intervals for each day of the week (thus, the intraday seasonal factor is different for each day of the week). Cubic splines are then used to smooth the time-of-day function that displays the well-known inverted-U shape.

Some alternative procedures have also been applied in the literature. Tsay (2002) uses quadratic functions and indicator variables. Drost and Werker (2004) only include an indicator variable for lunchtime in the conditional mean duration because, for their data, trading intensity except for lunchtime seems almost constant. Dufour and Engle (2000a) include diurnal dummy variables in their vector autoregressive system but fail to identify any daily pattern, except for the first 30 minutes of the trading day that appear to have significantly different dynamics from the rest of the data.[40] Veredas, Rodriguez-Poo, and Espasa (2001) propose a joint estimation of the two components of durations, dynamics and seasonality with the dynamics specified by (Log)-ACD models and the seasonality left unspecified and therefore estimated nonparametrically.

Despite the need for some caution already expressed in the literature (see, for example, Bauwens et al., 2004; Meitz and Teräsvirta, 2006, among others), the

[40] Note also that it is a current practice in studies on irregularly time-spaced data to remove the transactions during the first minutes of the trading session (the opening trades).

impact of the deseasonalisation procedure on the results has been insufficiently investigated and should receive further consideration. It is noteworthy that eliminating intraday seasonality does not affect the main properties of the durations discussed below.

1.3.2 Applications to trade durations

The most common type of event considered for defining financial durations is a trade. Thus, trade durations are simply the time intervals between consecutive transactions. Applications of ACD models to trade durations have been reported in numerous papers (see, among others, Engle and Russell, 1998; Jasiak, 1998; Engle, 2000; Zhang, Russell, and Tsay, 2001; Bauwens and Veredas, 2004; Manganelli, 2005; Bauwens, 2006). Generally, some authors were interested in the model specification necessary to fit the dynamics of different data while others focused on the testing of various market microstructure predictions.

1.3.2.1 Stylized facts of trade durations and model specification

Several stylized properties of trade durations that motivate the use of the ACD methodology have been identified in the empirical literature. First, all the papers report the phenomenon of clustering of trade durations, i.e., long (short) durations tend to be followed by long (short) durations that may be due to new information arising in clusters. This is also evidenced by the highly significant (positive) autocorrelations that generally start at a low value (around 0.10). The autocorrelation functions then decay slowly, indicating that persistence is an important issue when analyzing trade durations. Ljung-Box Q-statistics are often used to formally test the null hypothesis that the first (15 or 20) autocorrelations are 0. The statistics take very large values clearly indicating the presence of ACD effects, that is, duration clustering, at any reasonable level.[41] A slowly decaying autocorrelation function may be associated with a long-memory process,

[41] An alternative test for ACD effects has been proposed by Duchesne and Pacurar (2005) using a frequency domain approach.

which motivated Jasiak (1998) to introduce the FIACD model. Evidence for long memory has been constantly reported for IBM trade durations, for instance (see, for example, Engle and Russell, 1998; Jasiak, 1998; Bauwens et al. 2004).

Second, the majority of papers report that trade durations are overdispersed, i.e., the standard deviation is greater than the mean.[42] This can be tested formally using the dispersion test introduced by Engle and Russell (1998) or a Wald test for the equality of the first two sample moments, as in Dufour and Engle (2000a). Overdispersion is also apparent from the examination of the kernel densities of different trade durations that have a hump at very small durations and a long right tail (see, for instance, Bauwens et al., 2004; Bauwens and Giot, 2001; Engle and Russell, 1998).[43] This suggests that exponential distribution is not appropriate for unconditional distribution of trade durations (which does not mean however that conditional durations cannot be exponentially distributed).

Third, papers using transaction data reveal the existence of a large number of zero trade durations in the samples used (i.e., about two thirds of the observations for the IBM dataset originally used by Engle and Russell, 1998). Since the smallest time increment is a second, orders executed within a single second have the same time stamp. Following the original work of Engle and Russell (1998), the most common approach for dealing with zero durations consists of aggregating these simultaneous transactions. An average price weighted by volume is generally computed (when the variable price is also of interest), and all other transactions with the same time stamp are discarded. This procedure uses the microstructure argument that simultaneous observations correspond to split-transactions, that is, large orders broken into smaller orders to facilitate faster execution. When transactions do not have identical time stamps, identification

[42] Bauwens (2006) finds underdispersion for two out of four stocks considered from the Tokyo Stock Exchange. He eventually explains it by poor measurement of the very small durations of these stocks. Ghysels and Jasiak (1998) also report underdispersion for IBM trade durations computed with one month of data.

[43] One could argue that the hump close to the origin is an artifact of estimating the density of a positive variable using the kernel method. Therefore, most authors use the gamma kernel proposed by Chen (2000) and designed for this context.

of split-transactions may be more challenging. Grammig and Wellner (2002) consider as part of a large trade on the bid (ask) side of the order book only those trades with durations between sub-transactions of less than one second and whose prices are non-increasing (non-decreasing). Multiple transactions may, however, be informative as they reflect a rapid pace of the market. Zhang, Russell, and Tsay (2001) find that the exact number of multiple transactions does not carry information about future transaction rates, but the occurrence of multiple transactions does. Therefore, they incorporate this information into their TACD model through a lagged indicator variable for multiple transactions included as regressor in the conditional mean of durations. Bauwens (2006) artificially sets the duration between simultaneous trades at one second but we should mention the peculiar feature of his dataset from the Tokyo Stock Exchange in which two orders executed within two seconds have the same time stamp. An alternative explanation for zero durations is put forward by Veredas, Rodriguez-Poo, and Espasa (2001) based on the empirical observation of zero durations being clustered around round prices. Therefore, they argue that multiple transactions may occur because of many traders posting limit orders to be executed at round prices. Simply removing the simultaneous transactions certainly affects the dynamical properties of the trade durations. For instance, Veredas, Rodriguez-Poo, and Espasa (2001) and Bauwens (2006) report higher Q-statistics and residual autocorrelation, respectively, when zero durations are removed from the data. The parameters of the distribution of the innovation may also be altered. We believe that the literature still needs to explore alternative ways for dealing with zero durations as their impact is not sufficiently known.

It is noteworthy that a common practice when analyzing intraday durations consists of eliminating all interday (overnight) durations because they would distort the results. However, some alternative approaches have also been proposed. Manganelli (2005) treats the overnight period as if it was non-existent while Dufour and Engle (2000a) account for interday variations by including a dummy variable for the first observation of the trading day.

With regard to the model specification, most empirical applications of ACD models to trade durations have adopted linear ACD or Log-ACD specifications with low orders of lags, i.e., (1,1) or (2,2)-models, despite the findings discussed in Section 1.2 pointing out the need for nonlinear models. Further empirical evidence provided by Fernandes and Grammig (2006) shows that the problem of overpredicting short durations first identified by Engle and Russell (1998) can be palliated by allowing for a concave shocks impact curve.

Interestingly, several authors (Engle and Russell, 1998; Engle, 2000; Zhang, Russell, and Tsay, 2001; Fernandes and Grammig, 2006 among others) reveal substantial difficulties in completely removing dependence in the residual series, which suggests that the question of the most appropriate model for trade durations is far from being answered.[44] Bauwens et al. (2004) also report that none of the thirteen models considered (including ACD, log-ACD, TACD, SCD, and SVD models with various distributions for the innovation) provides a suitable specification for the conditional duration distribution. On the other hand, Dufour and Engle (2000a) find that the choice of the conditional distribution of durations apparently does not affect the out-of-sample predictions of the ACD model at short or longer horizons (but it becomes crucial when forecasting the whole density).

Many studies found evidence of high persistence of trade durations, the sum of the autoregressive coefficients (i.e., $\alpha + \beta$) being close to one while still in the stationary region (see, among others, Engle and Russell, 1998; Jasiak, 1998; Engle, 2000; Dufour and Engle, 2000a; Bauwens and Veredas, 2004).

However, few formal comparisons of existing ACD models have been made to date. Generally, studies limit to comparisons of a new proposed specification for the ACD model to the original Engle and Russell (1998)'s model.

Further investigation is imperatively needed as to the choice of the best model for different markets.

[44] A similar result is reported by Taylor (2004) for a sample of durations between non-zero price impact trades on the FTSE 100 index futures market.

1.3.2.2 Tests of market microstructure hypotheses As we have seen, the setup presented in Section 1.2 is very general and encompasses a variety of models. The autoregressive structure of the conditional duration function allows description of the duration clustering observed in intraday data, but this clustering phenomenon is not uniquely an empirical fact. The market microstructure literature offers some explanations for the autocorrelation of the duration process. However, the relationship between the literature on market microstructure and ACD models is reciprocally beneficial, the former providing theoretical explanations, the latter offering empirical support. An exhaustive review of market microstructure models is well beyond the scope of our paper and we, therefore, refer the interested reader to the excellent surveys of O'Hara (1995) or Madhavan (2000). Here, we shall limit our discussion to those theoretical models that provide insights on the use of the ACD framework and whose predictions have been empirically tested using ACD models.

Market microstructure is concerned with the study of the trading process and, as stated by O'Hara (1995, p.1), its research is "valuable for illuminating the behavior of prices and markets." Traditionally, price formation has been explained in the context of inventory models that focused on uncertainties in the order flow and the market maker's inventory position.[45] However, over the last several years, another branch of microstructure models has become more popular: information-based models that bring into play elements from asymmetric-information and adverse selection theory.[46] These models recognize the existence of different degrees of information in the market. Typically, two categories of traders are considered: informed and uninformed traders. Informed traders are assumed to possess private information and trade to take advantage of their superior information. Non-informed traders or liquidity traders trade for exogenous reasons, such as liquidity needs. The market maker or the specialist loses when trading with informed traders and has to compensate for

[45] See, for instance, Garman (1976), Stoll (1978), Ho and Stoll (1981), among others.

[46] The basic ingredients of information-based models are attributed to Bagehot (1971).

these losses when trading with the uninformed traders, which explains the bid-ask spread. The critical aspect in these models is that uninformed traders may learn by observing the actions of informed traders, or put differently, informed traders disclose information through their trades. Thus, prices reflect all publicly available information but private information is also progressively revealed by observing the actions of the informed traders. Information may be conveyed through various trade characteristics, such as timing, price, and volume. Several information-based models try to explain the complex relationships between these microstructure variables.

Among the key variables considered, the timing of trades plays an important role. Whereas initial microstructure models ignored the role of the time in the formation of prices, starting with the models of Diamond and Verrechia (1987) and Easley and O'Hara (1992), traders may learn from the timing of trades, hence the usefulness of ACD models for empirical investigations of trade durations.

Duration clustering is theoretically attributable to the presence of either informed traders or liquidity traders. According to the Easley and O'Hara (1992) model, informed traders only trade when new information enters the market while liquidity traders are assumed to trade with constant intensity. Thus, duration clustering occurs after information events because these increase the number of informed traders. Admati and Pfleiderer (1988) provide a different explanation of duration clustering. Their model distinguishes between two types of uninformed traders in addition to informed traders. Non-discretionary traders are similar to liquidity traders in the previous model whereas discretionary traders, while uninformed, can choose the timing of their trades. The authors show that the optimal behavior is a clumping behavior: discretionary traders select the same period for transacting in order to minimize the adverse selection costs and informed traders follow the pattern introduced by the discretionary traders.

In addition to providing theoretical explanations of the duration clustering phenomenon, market microstructure models make several predictions about the relationships between trade durations and other variables of the trading process.

Testable hypotheses linked to trade durations and empirical results From the papers we surveyed on trade durations, it appears that implications of the following main information-based models have been repeatedly tested using the ACD framework. These implications, sometimes contradictory, typically refer to the informational content of trade durations and the relationship between trading intensity and information-based trading.

Diamond and Verrechia (1987) use a rational expectation model with short-sale constraints. The main implication for empirical purposes is that when bad news enters the market, informed traders who do not own the stock cannot short-sale it because of the existing constraints. Hence, long durations are associated with bad news and should lead to declining prices.

In the Easley and O'Hara (1992) model, since informed traders trade only when there are information events (whether good or bad) that influence the asset price, short trade durations signify news arrival in the market and, hence, increased information-based trading. Consequently, the market maker needs to adjust his prices to reflect the increased uncertainty and risk of trading with informed traders, which translates into higher volatility and wider bid-ask spreads.[47]

Opposite relations between duration and volatility follow from the Admati-and-Pfleiderer model (1988) where frequent trading is associated with liquidity traders. Low trading means that liquidity (discretionary) traders are inactive, which leaves a higher proportion of informed traders on the market. This translates into higher adverse selection cost and higher volatility. Because of the lumping behavior of discretionary traders at equilibrium, we should also observe a clustering in trading volumes. Similarly, in the Foster and Viswanathan (1990) model the possibility of discretionary traders delaying trades creates patterns in trading behavior.

[47] Glosten and Milgrom (1985) first noted that when the probability of informed trading increases, the spreads become wider but time is not considered in their analysis.

Empirical tests of these predictions have been reported by several authors. Engle (2000) applies several specifications of the UHF-GARCH model to IBM data. He finds a statistically significant negative relation between durations (expected durations) and volatility as expected from Easley and O'Hara (1992). Interestingly, the coefficient of the current duration in the mean equation is negative and, hence, consistent with the Diamond and Verrechia (1987) model, i.e., long durations will lead to declining prices. When a dummy variable for large lagged spreads is included in the variance equation it has a positive sign: as predicted by Easley and O'Hara (1992), wider spreads predict rising volatility.

Whereas in the Engle model (2000), volatility does not impact trading intensity, Grammig and Wellner (2002) study the interaction of volatility and trading intensity by specifically modeling the intraday interdependence between them. They apply the IDV model to secondary market trading after the Deutsche Telekom IPO in November 1996.[48] Their result is consistent with the Admati-and-Pfleiderer model (1988): lagged volatility which is conceived as an indicator of informed trading[49] has a significantly negative impact on transaction intensity.

When applying their TACD model to IBM data, Zhang, Russell, and Tsay (2001) find that the fast trading regime is characterized by wider spreads, larger volume, and higher volatility, all of which proxy for informed trading. Thus, the results are consistent with Easley and O'Hara (1992) model but also with Easley and O'Hara (1987) who suggest that the likelihood of informed trading is positively correlated with trading volume. Even if their analysis is limited to one stock, an interesting finding that deserves attention is the fact that fast and slow trading regimes have different dynamics, which may suggest that the results observed on frequently traded stocks are not necessarily valid for less frequently traded stocks.

To examine the relationship between trading intensity and intraday volatility,

[48] Their interest in this event is motivated by the assertion made in the corporate finance literature that there is a large asymmetry of information in the market in the case of an IPO.

[49] See French and Roll (1986).

Feng, Jiang, and Song (2004) regress the realized volatility computed over 30-second time intervals against the forecast of trade duration based on the SCD models estimated with and without leverage (see Section 1.2). The models are applied to data for IBM, Boeing, and Coca Cola stocks. The results of the regression are consistent with the model of Easley and O'Hara (1992) as a significantly negative relation between trade durations and volatility is found for all three stocks.

Russell and Engle (2005) estimate a five-state ACM(3,3)-ACD(2,2) model on data for the Airgas stock traded on NYSE. They find that long durations are associated with falling prices, which is consistent with the predictions of Diamond and Verrechia (1987), and that the volatility per unit time is highest for short durations, as predicted by Easley and O'Hara (1992).

Another appealing approach for testing market microstructure hypotheses consists of using a VAR model that integrates several of the economic variables of interest. Dufour and Engle (2000b) analyze the price impact of trades in a large sample of 18 NYSE frequently traded stocks, using a bivariate 5-lags VAR-model for returns and trade sign in which the coefficients vary with the trading frequency as measured by trade durations. Their results show that shorter trade durations induce stronger positive autocorrelations of signed trades and larger quote revisions. For example, when a buy order is executed right after a previous order, its price impact is higher than that of a buy order arriving after a long time interval and also it becomes more likely to be followed by another buy order. When these results are associated with those of Hasbrouck (1991), it follows that short durations are associated with large spreads, large volumes, and high price impact of trades, consistent with Easley and O'Hara (1992) predictions.

More recently, other extensions of the Dufour and Engle VAR approach (2000b) have been proposed in the literature to include other variables such as the trade size and volatility. Spierdijk (2004) studies five frequently traded stocks from NYSE and reports similar results to Dufour and Engle (2000b). Moreover, she finds that large trades increase the speed of trading, while large returns decrease the trading intensity. Volatility is found to be higher when durations are short. Some of these results are also reported by Manganelli (2005) for a sample of 10 stocks from the NYSE. In particular, high volume is associated with increased volatility as predicted by Easley and O'Hara (1987): for example, larger trade sizes are more likely to be executed by informed traders and, hence, have a greater price impact. Also, durations have a negative impact on volatility that is consistent with Easley and O'Hara model (1992). An important element of Manganelli's study is the separate analysis of two groups of stocks classified according to their trading intensity. This is the first empirical study providing concrete evidence of a significant difference in dynamics between frequently traded and infrequently traded stocks. The usual relationships between duration, volume, and volatility are not empirically confirmed for his sample of infrequently traded stocks. It is also noteworthy that the empirically observed clustering in trading volumes (which is motivated theoretically by authors such as Foster and Viswanathan, 1990) is modeled by Manganelli (2005) by an Autoregressive Conditional Volume model similar to the ACD model.

Interestingly, all the papers mentioned above studied the equity market. Recently, Holder, Qi, and Sinha (2004) investigated the price formation process for the futures market. An approach similar to that of Dufour and Engle (2000b) is applied to transaction data for Treasury Note futures contracts traded at the Chicago Board of Trade. The analysis also includes the number of floor traders and the trading volume as explanatory variables. The major findings illustrate notable differences from the equity market. In particular, trade durations are significantly positively related to subsequent returns and the sign of trades.

Summing up, some remarks are noteworthy here. First, while all studies agree that trade durations have an informational content, the empirical results regarding the relationship between different trade variables are partially contradictory, in a way similar to theoretical microstructure models. Most of the papers seem to suggest that high trading periods are associated with high volumes and high volatilities and are due to the increased presence of informed traders in the market.

Second, the majority of papers reviewed use transaction data for very liquid blue chip stocks while a few studies suggest that the information dissemination for less frequently traded stocks might be quite different.

Finally, most of the results presented are obtained for the NYSE, a market that combines features of a price-driven market (presence of a market maker) with those of an order-driven market (existence of an order book). However, the learning process might be different in a pure order-driven market where no official market maker exists. Examinations of more markets/stocks would be valuable for a better understanding of the differences across market structures.

1.3.3 Applications to price durations

Instead of focusing the analysis on the arrival times of all transactions, examination of the arrival of particular events, such as a certain change in the price or the time necessary for trading a given amount of shares, may be needed. In the point process literature, retaining only the arrival times that are thought to carry some special information is called thinning the point process. Mostly used examples include price durations and volume durations.

First introduced in Engle and Russell (1997), price durations represent the times necessary for the price of a security to change by a given amount, C. The first point of the new thinned process usually corresponds to the first point of the original point process, $\tau_0 = t_0$. Then, if p_i is the price associated with the transaction time t_i, the series of price durations is generated by retaining all points i from the initial point process, $i > 1$, such that $|p_i - p_{i'}| \geq C$ where $i' < i$ is the index of the most recently selected point. To avoid the problem of the bid-ask bounce, Engle and Russell (1997, 1998) recommend defining price durations on the midprice of the bid-ask quote process.

The importance of price durations comes from their close relationship with the instantaneous volatility of price. As formally shown by Engle and Russell (1998), instantaneous intraday volatility is linked to the conditional hazard of price durations:

$$\sigma^2\left(t|\mathcal{F}_{i-1}\right) = \left(\frac{C}{P(t)}\right)^2 h\left(x_i|\mathcal{F}_{i-1}\right) \tag{1.26}$$

where $\sigma^2\left(t|\mathcal{F}_{i-1}\right)$ is the conditional instantaneous volatility , $P(t)$ is the the midquote price, and $h\left(x_i|\mathcal{F}_{i-1}\right)$ is the conditional hazard of price durations defined for the threshold C. Consequently, if an EACD model is applied on price durations $h\left(x_i|\mathcal{F}_{i-1}\right) = 1/\psi_i$.

1.3.3.1 Stylized facts of price durations and model specification

Studies investigating the empirical properties of price durations found the same characteristics that motivated the use of ACD models for trade durations: positive autocorrelations, overdispersion, a right-skewed shape, and strong intraday seasonality (see Engle and Russell, 1998; Bauwens and Giot, 2000; Bauwens and Veredas, 2004; Bauwens et al., 2004, Fernandes and Grammig, 2006, among others). As price durations are inversely related to volatility, seasonal patters of price durations may also be interpreted as intraday patterns of volatility.

Interestingly, price durations appear to be easier to model with regard to fully eliminating the serial dependence of residuals through the use of an ACD model (Bauwens et al., 2004; Fernandes and Grammig, 2006).

With regard to model specifications, a limited number of comprehensive comparisons exist and their results are rather contradictory. Bauwens et al., (2004) report that less complex ACD models such as the ACD and the Log-ACD outperform more complex models like the TACD, SVD and SCD, as long as they are based on flexible innovation distributions, such as the generalized gamma and the Burr distribution. The use of non-monotonic hazard functions for price durations is also advocated by Grammig and Maurer (2000). On the other hand, Fernandes and Grammig (2006) found evidence against standard specifications and recommend the logarithmic class of their AACD family for increased flexibility. Again, more systematic investigations would be interesting.

1.3.3.2 Tests of market microstructure hypotheses and volatility modeling The link of price durations to the volatility process makes the use of ACD models very appealing for testing market microstructure predictions in a very simple way. It is sufficient to add additional explanatory variables linked to different market characteristics into equation (1.4) for the conditional expected duration, and this also allows for a better model specification. In this respect, Log-ACD models have become very popular for they avoid the non-negativity constraints on parameters. Among the most relevant variables for investigating market microstructure effects, the lags of the following are typically used in analyses of stock markets (Engle and Russell, 1998; Bauwens and Giot, 2000; Bauwens and Giot, 2003; Bauwens and Veredas, 2004):

- The trading intensity: It is defined as the number of transactions during a price duration, divided by the value of this duration. According to Easley and O'Hara (1992), an increase in the trading intensity following an information event should be followed by shorter price durations as the market maker revises his quotes more frequently. A negative coefficient is found in empirical applications consistent with these predictions.

- The spread: In the Easley and O'Hara (1992) model, a high spread is associated with short durations, which is confirmed in empirical results by the negative coefficient of the lagged spread.

- The average volume per trade: Following implications of information-based models, a similar negative relationship between traded volume and price durations is typically found as a higher volume is indicator of informed trading and, hence, higher volatility.

Investigations of various relationships between market microstructure variables using ACD models for price durations have been also reported for derivatives markets (see Taylor, 2004 and Eom and Hahn, 2005 for the futures and option markets, respectively.)

An interesting application of ACD models for price durations has been proposed by Prigent, Renault, and Scaillet (2001). Building on a traditional binomial option pricing model, they relax some of its rigid assumptions by means of dynamic specifications. First, instead of considering fixed time intervals between price variations (jumps) of constant size, they employ a Log-ACD model for specifying the arrival times. Second, they model the probabilities of up and down moves by an ACM model with two states (that is, an Autoregressive Conditional Binomial model).

Starting from equation (1.26), ACD models for price durations can also be used as an alternative to standard GARCH models. This idea has been further explored by Giot (2000), Gerhard and Hautsch (2002), Kalimipalli and Warga (2002), and Giot (2002). Giot (2000) uses the estimated coefficients of an Log-ACD model applied to IBM durations for computing the intraday volatility directly from (1.26). Interesting insights come from the analysis of Gerhard and Hautsch (2002) on the LIFFE Bund future market. Their model, however, is a non-dynamic proportional intensity model for categorized durations including censoring effects due to market closure, and it does not belong to the ACD family. Kalimipalli and Warga (2002) use ACD-based volatility estimates as an explanatory variable in an ordered probit model for investigating the relationship between spreads, volatility, and volume for the ten most actively traded bonds on the Automated Bond System market maintained by NYSE. A statistically significant positive relationship between volatility and spreads is identified but, interestingly, a negative relationship between volume and spreads is found. According to Harris and Raviv (1993), this may be due to a lack of consensus among traders, therefore placing limit orders on both sides of the bid-ask spread. It also suggests a weak adverse-selection component of the spreads. Moreover, similar results are obtained when using GARCH-based volatilities, which confirms the robustness of the results.

Volatility is also an essential ingredient of risk management. Therefore, it is not surprising that its estimation based on price durations has finally been considered for constructing intraday Value at Risk (VaR) models. Giot (2002) quantifies the market risk based on intraday returns in a conditional parametric VaR framework. Three ARCH-type models are applied to the equidistantly time-spaced observations re-sampled from the irregularly time-spaced data, and a Log-ACD model serves to compute the volatility based on price durations. However, the results from the Log-ACD model are not completely satisfactory, the price durations based model failing most of the time for all the stocks considered[50]. As the author notes, possible explanations of this result may be found in the assumption of normality of intraday returns and the need of complicated time transformations to switch to the regularly time-spaced. It is noteworthy, however, that investigations of the benefits of using tick-by-tick data for risk management have only just started. An alternative approach has been recently proposed by Dionne, Duchesne, and Pacurar (2005) based on an ACD-GARCH model within a Monte Carlo simulation framework.

1.3.4 Applications to volume durations and other economic events

Another way of thinning a point process generates volume durations defined as the times until a given aggregated volume is traded on a market. Volume durations were introduced by Gouriéroux, Jasiak, and Le Fol (1999) as reasonable measures of liquidity that account simultaneously for the time and volume dimensions of the trading process. According to the conventional definition of liquidity (Demsetz, 1968; Black, 1971; Glosten and Harris, 1988), an asset is liquid if it can be traded as fast as possible, in large quantities, and with no significant impact on the price. Consequently, while neglecting the price impact of volumes, volume durations may still be interpreted as the (time) cost of liquidity.

[50] The performance of each model, including the Log-ACD based model, is assessed in a regularly time-spaced framework by computing its failure rate as the number of times returns are greater than the forecasted VaR.

There are much fewer studies applying ACD models to volume durations than to trade and price durations. Their main objective typically consists of finding the appropriate ACD specifications (Bauwens et al., 2004; Bauwens and Veredas, 2004; Veredas, Rodriguez-Poo, and Espasa, 2005; Fernandes and Grammig, 2006). These papers report very different statistical properties of volume durations compared to trade and price durations. Initial autocorrelations are larger and the density of volume durations appears as clearly hump-shaped. While trade and price durations are overdispersed, volume durations exhibit underdispersion, that is, the standard deviation is smaller than the mean.

With regard to model specification, Bauwens et al. (2004) find that ACD and Log-ACD models based on the Burr or generalized gamma distribution are useful for modeling not only price durations but volume durations as well. As expected, the EACD and SVD models perform badly on volume durations as the exponential and Pareto distribution, respectively, cannot describe the hump shape of the unconditional distribution of volume durations.

In recent years, the ACD model has also been applied to other irregularly time-spaced financial data that are not linked to the intraday trading process. For example, Fischer and Zurlinden (2004) examine the time spacings between interventions by central banks on foreign exchange markets. Using daily data on spot transactions of the Federal Reserve, the Bundesbank, and the Swiss National Bank on the dollar market, the authors conclude that traditional variables of a central bank's reaction function for interventions (for example, the volume and direction of the intervention) do not improve the ACD specification in their sample.

An interesting application of the ACD methodology has been suggested by Christoffersen and Pelletier (2004) for backtesting a VaR model. The main idea is very simple: if one defines the event where the ex-post portfolio loss exceeds the ex-ante predicted VaR as a violation, the clustering of violations can be described by an ACD model. Under the null hypothesis that the VaR model is correctly specified for a confidence level $1 - p$, the conditional expected duration until the next violation should be a constant equal to $1/p$ days which can be tested in several ways.

An attempt to apply ACD models to credit risk analysis can be found in Focardi and Fabozzi (2005) where defaults in a credit portfolio follow a point process. By using an ACD specification for the arrival times of defaults in a portfolio, one may estimate the aggregate loss directly without the need for modeling individual probabilities of defaults while accounting for the credit-risk contagion phenomenon through the clustering of the defaults. While conceptually interesting, this approach still has to be validated empirically, the authors investigating it only through simulations.

Given the wealth of possible specifications of the ACD models, we expect further applications of the ACD approach to other irregularly time-spaced events in the near future.

1.4 Conclusion

In this paper we have reviewed the theoretical and empirical literature on ACD models. Since its introduction by Engle and Russell (1998), several articles applying this class of models have already appeared. The motivation behind the use of ACD models is twofold. On the one hand, transaction data, increasingly available at a low cost over the last years, are irregularly time-spaced so that new econometric techniques are needed to deal with this feature. On the other hand, recent market microstructure models based on asymmetric-information theory argue for the role of time in the dissemination of information among different participants in the trading process.

As our survey shows, much progress has been realized in understanding ACD models. Initial research has been oriented towards proper modeling of the data at hand and has focused naturally on palliating some of the inconveniencies of the original model. Extensions have been undertaken simultaneously on two fronts:

developing more flexible specifications of the conditional mean and looking for more flexible distributions of the error term. Importantly, the expansion of ACD models has been fueled by the strong similarity between the ACD and GARCH models which impacted the search for new specifications and estimation methods. Compared with the GARCH literature, however the current ACD literature can be considered rather young and not as rich. We believe that ACD modeling could nevertheless benefit by freeing itself from the GARCH influence.

Given the increasing variety of existing specifications, one would like to know which model is the most appropriate under specific circumstances. Very few studies compare different ACD models using the same data, and recent specifications are barely included. Typically, when a new type of ACD model has been proposed, its performance has been compared with that of the original ACD form of Engle and Russell (1998). In our opinion, extensive comparisons of several models using various evaluation criteria (both in-sample and out-of-sample) on different datasets would greatly benefit the applied econometrician. It is noteworthy that only the basic ACD and Log-ACD models are used in most of the empirical studies surveyed. Despite the evidence of nonlinearity reported by several authors, it is still not clear which type of nonlinear ACD model should be recommended for specific conditions. This same discrepancy between the theoretical and applied literature has been observed with regard to existing testing procedures for the ACD models. Even though several tests have been developed, their relative performance remains insufficiently investigated.

ACD models have been used for describing not only the durations between different market events, but also as a building block for jointly modeling duration and other market characteristics, such as duration and price. If initial interest focused on marginal models for the arrival times of events, joint models have quickly captured attention for the examination of several market microstructure theories. Multivariate models (for trades and quotes, for instance) may improve understanding of the complex relationships between several trade variables, such as price, volume, and volatility. As such, they may have implications for market designers and policy makers. As many studies use transaction data from NYSE, examination of data from other markets could help to evaluate the differences between trading systems regarding the price discovery process.

A common assumption when working with joint models for durations and marks is that of the exogeneity of durations, often considered without any formal testing. We expect more work on this issue in the near future as warnings for caution have already been made.

Examination of existing empirical studies on ACD models revealed other problems that should be addressed in future research. Intraday deseasonalization techniques seem to have a significant but not very well understood effect on any model used. The treatment of zero-durations arising from observations with the same time stamp does not reflect a consensus among researchers, even if the dominant trend consists of simply eliminating them. A few studies have also pointed out substantial differences between the dynamics of actively traded stocks used mostly in empirical applications and those of infrequently traded stocks.

The class of ACD-GARCH models, as well as the link between price durations and instantaneous volatility, could also serve in developing risk measures based on transaction data and that are useful for agents very active on the market. While some work has already been initiated, we think that this issue will receive further consideration.

A promising line of research seems to be the application of ACD models to irregularly time-spaced data other than intraday data on the trading process. Finally, we think that the use of ACD models by applied researchers could be encouraged by the integration of some of the existing models and testing procedures in popular softwares.

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Chapter 2

On testing for duration clustering and diagnostic checking of models for irregularly spaced transaction data

2.1 Introduction

Since Engle and Russell (1997, 1998) introduced the Autoregressive Conditional Duration (ACD) model, there has been considerable interest in modelling high frequency financial data that arrive at irregular time intervals. Examples of such financial data include trade durations (quote durations), that is the times between consecutive trades (quotes). Another important concept concerns price durations, obtained by thinning the marked point process for the quotes with respect to a minimum change in the price. In applications, market participants may be interested by the time between quotes such that a given volume c, say $c = 90000$, of shares is traded. This example is called volume durations and leads usually to much smaller sample sizes than for trade durations. In practice, they are computed by thinning the trade process or the quote process such that the retained durations are characterized by a total traded volume of at least c. See Bauwens and Giot (2001) for details.

The ACD model treats the arrival time intervals between events of interest (e.g., trades, quotes, price or volume durations, among others), as a nonnegative stochastic process. It provides a model for the conditional duration between events; conditional durations may be expressed as a linear function of past durations and past conditional durations. Various generalizations of ACD models have been proposed in the literature. Nonlinear ACD models are discussed in the seminal work of Engle and Russell (1998). Bauwens and Giot (2000, 2003) considered the log-ACD model and the asymmetric ACD model. The threshold ACD model has been proposed by Zhang, Russell and Tsay (2001) to allow the expected duration to depend nonlinearly on past information variables. Fractionally integrated ACD models have been studied in Ghysels and Jasiak (1998a) and Jasiak (1999). Such models are useful in the presence of highly persistent duration clustering. Ghysels and Jasiak (1998b) proposed the so-called ACD-GARCH model, obtained by considering simultaneously the GARCH models for the volatility and the ACD models for the durations. Grammig and Maurer (2000) studied the ACD model based on the Burr distribution for the innovation which includes the exponential and Weibull distribution as special cases. Specification tests of the innovation distribution were proposed by Engle and Russell (1998) who check the first and second moments of the residuals with a particular attention to measure excess dispersion; Bartlett identity tests were developed in Prigent, Renault and Scaillet (2001) and the QQ-plots were considered in Bauwens and Veredas (2004). Recently, Fernandes and Grammig (2005) developed another approach for testing the innovation distribution of ACD specifications, by gauging the distance between the parametric density and hasard rate functions implied by the duration process and their non-parametric estimates. Another testing framework for financial duration models is the density forecast evaluation technique used by Bauwens, Giot, Grammig and Veredas (2004). Comprehensive introductions of ACD models are provided in Tsay (2002) and Engle and Russell (2006).

A critical step before trying to estimate a particular model for the conditional duration is to test for duration clustering, what we call ACD effects, which are structurally similar to test for autoregressive conditional heteroscedastic (ARCH) effects. To this end, we must verify if there is evidence of duration clustering in the arrival times. This is a sound practice before trying to adjust a particular model. Commonly used tests for ACD effects are the portmanteau test statistics of Box and Pierce (BP) (1970) or Ljung and Box (LB) applied to the raw durations (see, e.g., Engle and Russell (1997, 1998), Bauwens and Giot (2001), among others). Li and Yu (2003) derived the asymptotic distribution of the residual autocovariances in ACD models. Hong and Lee (2003) suggested an omnibus procedure which can be used as a misspecification test for ACD models, based on the generalized spectral density. Meitz and Teräsvirta (2004) studied Lagrange multiplier tests for adjusting ACD models. In this paper, we advocate the use of the classical spectral density, which is capable of describing a signal at various frequencies. This tool has been widely accepted and used in engineering and applied mathematics (see, e.g., Priestley (1981)). By adapting the test statistics of Hong (1996, 1997) in the ACD framework, we propose two classes of test statistics for ACD effects. More precisely, Hong test statistics are constructed using regression residuals, with known zero mean. Here, the procedures for ACD effects are based on raw duration data, and we establish the asymptotic distributions of the test statistics when the mean of the duration data needs to be estimated. The tests of the first class rely on a comparison of a kernel-based normalized spectral density estimator and the normalized spectral density under the null hypothesis of no ACD effects, using a particular norm. Examples of norms include L_2 norm, Hellinger distance and the Kullback-Leibler information criterion. Using the truncated uniform kernel and the L_2 norm, one member of the class provides a generalized BP test statistic. However, when the low order autocorrelations are large, and if the autocorrelations decay quickly to zero as a function of the order of lag, many kernels may give a higher power than the truncated uniform kernel. The distribution of the tests is asymptotically normal and the tests based on the L_2 norm, the Hellinger distance and the Kullback-Leibler information criterion are asymptotically equivalent under the null hypothesis, given certain conditions. The second class of test statistics for ACD effects exploits the one-sided nature of the problem. The BP/LB test statistics do not exploit such an one-sided nature. Our approach is similar in spirit to the spectral approach of Hong (1997) for testing for ARCH. See also Lee and King (1993). The tests rely on a kernel-based spectral density estimator evaluated at frequency zero. The resulting tests consist in a weighted sum of sample autocorrelations of the raw durations. As in the first class of tests, the weighting function typically gives more (less) weight to lower (higher) orders of lags. The asymptotic distribution of the test statistics in the second class is $N(0,1)$ under the null hypothesis of no ACD effects. A natural question concerns which class should be preferred. Asymptotic arguments suggest that the tests in the first class might be more powerful than the tests in the second class asymptotically. However, to exploit the one-sided nature of the alternative hypothesis may be powerful in small samples. We explore this issue in our simulation experiments.

The class of tests for the adequacy of an ACD model is obtained by comparing a kernel-based spectral density estimator of the estimated standardized residuals and the null hypothesis of adequacy using a norm. The proposed test statistics in this class possess an asymptotic normal distribution. We establish rigorously that parameter estimation has no impact on the distribution of the test statistics, asymptotically. With the L_2 norm and the truncated uniform kernel, we retrieve a generalized BP test statistic applied to the estimated standardized residuals. However, using a kernel different from the truncated uniform kernel, we may obtain more powerful test procedures in many practical situations. A technical merit of the paper is to establish, in the context of the adjustment of ACD models, the asymptotic distributions of the studied test statistics.

The organization of the paper is as follows. In Section 2.2, after some preliminaries, we describe the hypotheses of interest. In Sections 2.3 and 2.4 we present two classes of test statistics for ACD effects and one class of test statistics for the adequacy of an ACD model. We establish that the proposed test statistics have an asymptotic normal distribution under their respective null hypothesis. In a given class of tests, we give conditions under which the test statistics based on the considered norms are asymptotically equivalent. The asymptotic results in the ACD context represent the main technical achievements of the paper. In Section 2.5, we present some simulation results, including a level and a power study of the tests for ACD effects, and for the adjustment test statistics of ACD models. The proposed test statistics are compared with respect to levels and powers with many kernels, including uniform and non uniform weighting, to the BP/LB test statistics. The Section 2.6 contains an application with the IBM data considered by Engle and Russell (1998) for trade and volume durations. Finally, Section 2.7 concludes the paper. The proofs of the theorems are given in the Appendix.

2.2 Preliminaries and framework

2.2.1 Preliminaries

Suppose that the duration process $X = \{X_t, t \in \mathbb{Z}\}$ is a nonnegative stationary process such that

$$X_t = D_t^0 \epsilon_t, \tag{2.1}$$

where $\epsilon = \{\epsilon_t, t \in \mathbb{Z}\}$ represents a nonnegative, independently and identically distributed (iid) sequence with probability density $p_\epsilon(\cdot)$. We assume that the expectation of ϵ_t is one, that is $E(\epsilon_t) = 1$. The conditional duration $D_t^0 \equiv D^0(\mathcal{F}_{t-1}) = E(X_t|\mathcal{F}_{t-1})$ is supposed to be a nonnegative measurable function of $\mathcal{F}_{t-1}$, where $\mathcal{F}_{t-1}$ denotes the information set generated by all past observations up to and including the tth financial transaction.

Let $Y = \{Y_t, t \in \mathbb{Z}\}$ be an arbitrary second order stationary process whose mean is $E(Y_t) = \mu$. The autocovariance at lag j is given by $\gamma_Y(j) = E\{(Y_t - \mu)(Y_{t-j} - \mu)\}, \quad j \in \mathbb{Z}$ and the autocorrelation at lag j is defined by $\rho_Y(j) = \gamma_Y(j)/\gamma_Y(0)$. If $\sum_{j=0}^{\infty} |\gamma_Y(j)| < \infty$, the normalized spectral density of the process Y is given by

$$f_Y(\omega) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \rho_Y(h) e^{-i\omega h}, \;\; \omega \in [-\pi, \pi].$$

Let $Y_1, Y_2, \ldots, Y_n$ be a realization of length n of the process Y. The sample autocovariance at lag j, $0 \leq |j| \leq n-1$ is defined by $C_Y(j) = n^{-1} \sum_{t=j+1}^{n} (Y_t - \bar{Y})(Y_{t-j} - \bar{Y})$, $\;0 \leq |j| \leq n-1$, where $\bar{Y} = n^{-1} \sum_{t=1}^{n} Y_t$, and the corresponding sample autocorrelation is

$$R_Y(j) = C_Y(j)/C_Y(0), \;\; 0 \leq |j| \leq n-1. \tag{2.2}$$

The classical nonparametric kernel-based estimator of the spectral density $f_Y(\omega)$ of Y is given by

$$f_{Yn}(\omega) = \frac{1}{2\pi} \sum_{j=-n+1}^{n-1} k(j/p_n) R_Y(j) e^{-i\omega j}, \tag{2.3}$$

where $k(\cdot)$ is a kernel or a lag window. The parameter p_n corresponds to a truncation point when the kernel is of compact support or a smoothing parameter when the kernel is unbounded. The assumptions on the kernel are summarized as follows.

Assumption A:

(1) The kernel $k : \mathbb{R} \to [-1, 1]$ is a symmetric function, continuous at 0, having at most a finite number of discontinuity points, such that $k(0) = 1$ and $\int_{-\infty}^{\infty} k^2(z)dz < \infty$.

(2) $\int_{-\pi}^{\pi} |k(z)|dz < \infty$ and $K(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} k(z)e^{-iz\lambda}dz \geq 0$, $\lambda \in (-\infty, \infty)$.

Most commonly used kernels satisfy Assumption A(1). An example is the rectangular or truncated uniform kernel $k_T(z) = I[|z| \leq 1]$, where $I(A)$ denotes the indicator function of the set A. When $k = k_{TR}$, the estimator (2.3) reduces to the truncated periodogram. In Assumption A(2), the condition $\int_{-\pi}^{\pi} |k(z)|dz < \infty$ guaranties that the Fourier transform $K(\cdot)$ exists. This condition ensures that the kernel-based spectral density estimator is nonnegative. Consequently, Assumption A(2) rules out k_{TR}. Examples of kernels satisfying Assumption A include the Bartlett, Daniell, Parzen and Quadratic-Spectral kernels (see, e.g., Priestley (1981)).

Formula (2.2) allows us to construct the sample autocorrelation function of the raw durations, that we denote $R_X(j)$, $|j| \leq n-1$. Using formula (2.3), a spectral density estimator of the raw durations, noted $f_{Xn}(\omega)$, $\omega \in [-\pi, \pi]$, can be constructed. Once a particular model is estimated, the residuals $\hat{r}_t$ (say), $t = 1, \ldots, n$ are obtained. The sample autocorrelation function of the sample residuals is given by $R_{\hat{r}}(j)$, $|j| \leq n-1$ and $f_{\hat{r}n}(\omega)$, $\omega \in [-\pi, \pi]$ corresponds to the estimated spectral density of the estimated standardized duration residuals.

2.2.2 Autoregressive conditional duration models

Engle and Russell (1998) assumed that the durations admit the multiplicative representation (2.1). They proposed different specifications for the conditional duration D_t. A first functional form for D_t is the m-memory conditional duration process

$$D_t = \omega + \sum_{h=1}^{m} \alpha_h X_{t-h}, \tag{2.4}$$

denoted ACD(m), which depends on the m most recent durations. The constant m is a fixed integer. To ensure that D_t is strictly positive for all realizations of X_t, $t = 1, \ldots, n$, it is required that $\omega > 0$ and $\alpha_h \geq 0$, $h = 1, \ldots, m$. Another popular functional form for D_t discussed in Engle and Russell (1998) is the ACD(m,q) model given by

$$D_t = \omega + \sum_{h=1}^{m} \alpha_h X_{t-h} + \sum_{h=1}^{q} \beta_h D_{t-h}. \tag{2.5}$$

To ensure that D_t is strictly positive for all realizations of X_t, a sufficient condition is that $\omega > 0$, $\alpha_h \geq 0$, $h = 1, \ldots, m$ and $\beta_h \geq 0$, $h = 1, \ldots, q$. Noting the similarity between ACD models and GARCH models and using the results of Nelson and Cao (1992), it follows that these conditions are necessary and sufficient for an ACD(1,1) but they can be weakened for higher order ACD models.

In our framework, we do not make any particular distributional assumption on the probability density $p_\epsilon(\cdot)$ in (2.1). When $p_\epsilon(\cdot)$ is the exponential or Weibull distribution, Engle and Russell (1998) denote the resulting ACD model an EACD or WACD model, respectively. Grammig and Maurer (2000) consider the Burr distribution for $p_\epsilon(\cdot)$, which includes the EACD and WACD models as special cases. Tsay (2002) discusses the generalized gamma distribution as another possible choice for the distribution of ϵ_t; the resulting model is called the GACD model. When testing for the adequacy of ACD models, our main interest in this paper will be the model specification for D_t^0. By comparison, Fernandes and Grammig (2005) focused on specification tests for the distribution of ϵ_t.

Models (2.4) and (2.5) are special cases of the following general linear process for D_t:

$$D_t = \omega + \sum_{h=1}^{\infty} \alpha_h X_{t-h}, \tag{2.6}$$

where to ensure that D_t is strictly positive, we must impose that $\omega > 0$, $\alpha_h \geq 0$, for all $h = 1, 2, \ldots, \infty$. See Nelson and Cao (1992) for more details in the ARCH framework. This general linear process will appear useful for constructing test statistics which should have power for a large class of alternatives. In the next section, we use (2.6) to motivate test statistics for ACD effects.

2.3 Two classes of test statistics for testing for ACD effects

2.3.1 Tests based on a kernel-based spectral density estimator of the raw durations and a norm

In this section we develop two classes of test statistics for the existence of ACD effects. Under the general linear process (2.6), the null hypothesis of no ACD effects is given by

$$\mathbb{H}_{0,eff} : \alpha_h = 0, \text{for all } h > 0. \tag{2.7}$$

The alternative hypothesis that ACD effects are present is

$$\mathbb{H}_{1,eff} : \alpha_h \geq 0, \text{for all } h > 0, \text{with at least one strict inequality.} \tag{2.8}$$

In terms of the normalized spectral density of the raw durations, under $\mathbb{H}_{0,eff}$, we have that $f_X(\omega) = f_{X0}(\omega) \equiv 1/(2\pi)$, that is f_X is the flat spectrum under the null. However, under the alternative, the spectrum will not be equal to $1/(2\pi)$ in general under the linear process given by (2.6). We now introduce a distance measure $d(f_1; f_2)$ for two spectral densities, satisfying $d(f_1; f_2) \geq 0$ and $d(f_1; f_2) = 0$ if and only if $f_1 = f_2$ (note that we do not need the triangular inequality in our discussion). A test statistic for the null hypothesis of no ACD effects can be based on $d(f_{Xn}; f_{X0})$, where $f_{Xn}(\omega)$ corresponds to the normalized kernel-based spectral density estimator of the unknown spectral density $f_X(\omega)$. As a first example, the L_2 norm between f_X and f_{X0} is given by

$$Q^2(f_X; f_{X0}) = \pi \int_{-\pi}^{\pi} \{f_X(\omega) - f_{X0}(\omega)\}^2 d\omega.$$

Another possibility is the Hellinger distance defined by

$$H^2(f_X; f_{X0}) = 2 \int_{-\pi}^{\pi} \{f_X^{1/2}(\omega) - f_{X0}^{1/2}(\omega)\}^2 d\omega.$$

The Kullback-Leibler information criterion, called sometimes the relative entropy, provides another measure given by

$$I(f_X; f_{X0}) = -\int_{\Omega(f_X)} \log\{f_X(\omega)/f_{X0}(\omega)\} f_{X0}(\omega) d\omega,$$

where $\Omega(f_X) = \{\omega \in [-\pi, \pi] \mid f_X(\omega) > 0\}$ corresponds to the set where f_X is strictly positive. Each measure possesses its own merits. As we will show below, the L_2 norm delivers a computationally convenient test statistic. No numerical integration is needed in practice, since the test statistic based on this particular norm reduces to a weighted sum of squared sample autocorrelations. The weights depend on the chosen kernel $k(\cdot)$. Using the particular kernel $k = k_{TR}$ and considering $Q^2(f_{Xn}; f_{X0})$, we retrieve essentially the BP test statistic. The Hellinger distance corresponds to the L_2 norm between $f_X^{1/2}$ and $f_{X0}^{1/2}$. Contrary to

$Q^2(f_{Xn}; f_{X0})$, the Hellinger distance does not give the same weight to the difference between f_X and f_{X0}. When ACD effects are less persistent, the Hellinger distance gives more weight to small differences and less to large differences between f_X and f_{X0}; the resulting test statistic should be powerful in small samples. Finally, the Kullback-Leibler information criterion has an appealing information-theoretic interpretation.

We consider a class of test statistics for $\mathbb{H}_{0,eff}$, noted $\mathcal{E}(d; k; p_n)$, depending on a distance measure and a kernel function. A first member of the class based on the L_2 norm is given by

$$\begin{aligned} T_{\mathcal{E}1n} \equiv T_{\mathcal{E}n}(Q^2; k; p_n) &= \frac{nQ^2(f_{Xn}; f_{X0}) - K_{2n}(k)}{\{2K_{4n}(k)\}^{1/2}}, \\ &= \frac{n\sum_{j=1}^{n} k^2(j/p_n)R_X^2(j) - K_{2n}(k)}{\{2K_{4n}(k)\}^{1/2}}, \end{aligned}$$

where the last expression is obtained using the identity of Parseval and $K_{2n}(k) = \sum_{j=1}^{n-1}(1-j/n)k^2(j/p_n)$ and $K_{4n}(k) = \sum_{j=1}^{n-2}(1-j/n)(1-(j+1)/n)k^4(j/p_n)$. The quantities $K_{2n}(k)$ and $2K_{4n}(k)$ correspond essentially to the mean and variance of $nQ^2(f_{Xn}; f_{X0})$. A second member of the class based on the Hellinger's distance is given by

$$T_{\mathcal{E}2n} \equiv T_{\mathcal{E}n}(H^2; k; p_n) = \frac{nH^2(f_{Xn}; f_{X0}) - K_{2n}(k)}{\{2K_{4n}(k)\}^{1/2}},$$

and the third member in the class $\mathcal{E}$ is based on the Kullback-Leibler information criteria:

$$T_{\mathcal{E}3n} \equiv T_{\mathcal{E}n}(I; k; p_n) = \frac{nI(f_{Xn}; f_{X0}) - K_{2n}(k)}{\{2K_{4n}(k)\}^{1/2}}.$$

Our first result establishes the asymptotic distribution of the test $T_{\mathcal{E}1n}$ under the null hypothesis. The second part of the Theorem establishes the asymptotic equivalence between $T_{\mathcal{E}in}$, $i = 1, 2, 3$ under the null hypothesis. The symbol $\rightarrow_L$ stands for 'convergence in distribution'.

Theorem 1 *(a) Assume A(1), $E(X_t^4) < \infty$, $p_n \to \infty$ and $p_n/n \to 0$. Under*

$\mathbb{H}_{0,eff}$,

$$T_{\mathcal{E}1n} \rightarrow_L N(0,1).$$

(b) Assume the same hypotheses that in (a) with the more restrictive assumption $p_n^3/n \rightarrow 0$. *Furthermore, assume A(2). Under* $\mathbb{H}_{0,eff}$, $T_{\mathcal{E}1n} - T_{\mathcal{E}2n} = o_p(1)$; $T_{\mathcal{E}1n} - T_{\mathcal{E}3n} = o_p(1)$.

A simple application of Slutsky's lemma allows us to conclude that under the hypotheses of Theorem 1 (b), $T_{\mathcal{E}2n} \rightarrow_L N(0,1)$ and $T_{\mathcal{E}3n} \rightarrow_L N(0,1)$. Note that in Theorem 1 (b) the conditions on the kernel $k(\cdot)$ and on p_n are more restrictive than for $T_{\mathcal{E}1n}$.

When the kernel k is the truncated uniform kernel $k = k_{TR}$, we obtain

$$T_{\mathcal{E}1n}(Q^2; k_{TR}; p_n) = \frac{n \sum_{j=1}^{p_n} R_X^2(j) - p_n}{(2p_n)^{1/2}}.$$

Consequently, $T_{\mathcal{E}1n}(Q^2; k_{TR}; p_n)$ can be viewed as a generalized BP type test statistic. However, with a kernel different from k_{TR}, we expect more powerful procedures, specially when the true autocorrelation function decays to zero quickly as a function of the lag, with large autocorrelations when the orders of lags are small. Recognizing the similarities between testing for ARCH effects and testing for ACD effects (see Hong and Shehadeh (1999), among others), we expect more powerful procedures in many situations, by choosing a kernel giving more (less) weight to lower (higher) orders of lags. This is confirmed in our simulation results of the Section 2.5, where we compare empirically the test statistics based on different kernels and different distance measures.

2.3.2 Tests based on the spectral density evaluated at the zero frequency

The alternative hypothesis $\mathbb{H}_{1,eff}$ that ACD effects exist is one-sided. The test statistics which exploit the one-sided nature are expected to yield better power in small and moderate samples. However, it happens that the sample sizes of financial data can be quite large. For example, for the IBM data discussed in Engle and Russell (1998) and Engle (2000), the initial sample size was approximately $n = 60000$ observations. Each observation corresponded to a raw duration between two successive transactions, such that the retained durations were strictly positives. The power of the test statistics in such circumstances is probably of second importance if the test is consistent under the hypotheses of interest. However, the sample size was reduced to $n = 1347$ after thinning the data in studying price movements, which represents quite a large sample size reduction. Bauwens and Giot (2001) studied trade, price and volume durations for various stocks, including AWK, Disney, IBM and SKS. When studying volume durations, the sample size was in some cases less than $n = 300$. Consequently, to have powerful tests exploiting the one-sided nature of the alternative hypothesis $\mathbb{H}_{1,eff}$ seems highly desirable, particularly in such situations.

We propose an one-sided test statistic for ACD effects using a frequency domain approach. Like an ARCH process, an ACD process always possesses nonnegative autocorrelations at any lag, resulting in a spectral mode at frequency zero under, and only under, the alternative hypothesis $\mathbb{H}_{1,eff}$. This suggests to examine more precisely $f_X(0)$. The general linear process (2.6) implies that $X_t = \omega + \sum_{h=1}^{\infty} \alpha_h X_{t-h} + v_t$, where $v_t = X_t - D_t$ is a martingale difference sequence with respect to $\mathcal{F}_{t-1}$ by construction. This means that $E(v_t|\mathcal{F}_{t-1}) = 0$, a.s.. Under the null hypothesis $\mathbb{H}_{0,eff}$, $X_t = \omega + v_t$ is a white noise process. Thus, $f_X(0) = 1/(2\pi)$. Under the alternative hypothesis $\mathbb{H}_{1,eff}$, we have that $\rho_X(h) \geq 0, \forall h \neq 0$ and the inequality is strict for at least one $h \neq 0$. Consequently $f_X(0) > 1/(2\pi)$. This suggests to construct a test statistic based on the difference

$$f_{Xn}(0) - \frac{1}{2\pi}. \tag{2.9}$$

The presence of ACD effects is suggested if large differences are observed in (2.9).

An appropriate standardized version of (2.9) is given by

$$T_{\mathcal{E}4n}(k;p_n) = K_{2n}^{-1/2}(k)n^{1/2}\sum_{j=1}^{n-1}k(j/p_n)R_X(j).$$

The asymptotic distribution under the null hypothesis of $T_{\mathcal{E}4n}(k;p_n)$ is established in Theorem 2.

Theorem 2 *Assume A(1), $E(X_t^4) < \infty$ and $p_n/n \to 0$ as $n \to \infty$. Under $\mathbb{H}_{0,eff}$,*

$$T_{\mathcal{E}4n}(k;p_n) \to_L N(0,1).$$

Note that the asymptotic distribution is established when p_n is fixed or if $p_n \to \infty$ such that $p_n/n \to 0$. Our test statistic $T_{\mathcal{E}4n}(k;p_n)$ is similar to the one-sided test of Hong (1997), which is a powerful one-sided test statistic for ARCH effects. When the truncated kernel k_{TR} is used, the test statistic $T_{\mathcal{E}4n}(k;p_n)$ becomes $T_{\mathcal{E}4n}(k_{TR};p_n) = (p_n n^{-1})^{1/2}\sum_{j=1}^{p_n} R_X(j)$, which takes the same structure that the LBS test of Lee and King (1993) for ARCH effects. A natural question concerns which class of test statistics is preferable in practice. Following Hong and Shehadeh's (1999) approach, it follows that the test statistics $T_{\mathcal{E}in}(d;k;p_n)$, $i \in \{1,2,3\}$ should be more powerful than $T_{\mathcal{E}4n}(k;p_n)$ asymptotically, since asymptotic arguments demonstrate that $T_{\mathcal{E}in}(d;k;p_n)$, $i \in \{1,2,3\}$ can detect alternatives of order $O(p_n^{1/4}/n^{1/2})$, while $T_{\mathcal{E}4n}(k;p_n)$ can only detect alternatives of order $O(p_n^{1/2}/n^{1/2})$. However, when the sample size is rather small, it is expected that a test which exploits the one-sided nature of the alternative hypothesis should have a better power under certain conditions. We will compare empirically the power of the test statistics $T_{\mathcal{E}in}$, $i \in \{1,2,3,4\}$ in Section 2.5.

2.4 Class of test statistics for the adequacy of ACD models

2.4.1 Tests based on a kernel-based spectral density estimator of the estimated residuals and a norm

The first step in modelling financial durations is to test for ACD effects. If evidence of duration clustering is found, the practitioner may decide to formulate a certain ACD model to fit the data. A natural question concerns the adequacy of that model. An approach advocated in Engle and Russell (1997, 1998) and others, consists to examine whether the standardized duration residuals $\hat{r}_t = X_t/D(\mathcal{F}_{t-1}, \hat{\boldsymbol{\theta}}), t = 1, \ldots, n$ contain any remaining dependence structure, where $D(\mathcal{F}_{t-1}, \hat{\boldsymbol{\theta}})$ represents a model for the conditional duration. Engle and Russell (1997, 1998), Tsay (2002), Bauwens and Giot (2001), applied BP/LB test statistics based on the residuals $\{\hat{r}_t, t = 1, \ldots, n\}$.

More formally, suppose that a particular ACD model is adequate for the durations $\{X_t, t \in \mathbb{Z}\}$. Denote this model $D(\mathcal{F}_{t-1}, \boldsymbol{\theta}_0)$, where $\boldsymbol{\theta}_0$ represents the vector of unknown parameters. In such case, in the general model (2.1), we must have that $D_t^0 = D(\mathcal{F}_{t-1}, \boldsymbol{\theta}_0)$, *a.s.*. Consequently, $\epsilon_t = X_t/D(\mathcal{F}_{t-1}, \boldsymbol{\theta}_0)$ is an iid white noise process. Denote $r_t = X_t/D(\mathcal{F}_{t-1}, \boldsymbol{\theta})$, where $\boldsymbol{\theta} = \text{plim}\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\theta}}$ is a certain estimator of $\boldsymbol{\theta}_0$. When the ACD model is inadequate for X_t, we must have that $D_t^0 \neq D(\mathcal{F}_{t-1}, \boldsymbol{\theta})$ with strictly positive probability for all $\boldsymbol{\theta}$. When $\rho_r(j) \neq 0$ for at least one j different of zero, the process $r = \{r_t, t \in \mathbb{Z}\}$ will contain dependence structure and will not be iid. Consequently, in terms of the spectral density of the process r, the null hypothesis is given by

$$\mathbb{H}_{0,adj} : f_r(\omega) = f_{r0} \equiv 1/(2\pi), \tag{2.10}$$

since $r_t = \epsilon_t$ and $\{\epsilon_t, t \in \mathbb{Z}\}$ is an iid process. On the other side, the alternative hypothesis is

$$\mathbb{H}_{1,adj} : f_r(\omega) \neq 1/(2\pi), \tag{2.11}$$

since under the alternative hypothesis of inadequacy, if $\rho_r(j) \neq 0$ for at least one $j \neq 0$, the spectral density of r will be different from $1/(2\pi)$.

Note that under $\mathbb{H}_{1,adj}$, the standardized duration residuals could exhibit any departure from $1/(2\pi)$. Since positive and/or negative autocorrelations could occur, the alternative hypothesis is not one-sided. In an unified way, to test for model adequacy, we present a class of test statistics similar to the class presented in Section 2.3.1, based on the choice of a norm and of a kernel-based spectral density estimator of the standardized duration residuals. Consequently, we adapt Hong test statistics for serial correlation in the ACD framework, and we justify the asymptotic distributions of the resulting test statistics; these test statistics are then valid for adjusting ACD models.

The class of test statistics for $\mathbb{H}_{0,adj}$, noted $\mathcal{A}(d;k;p_n)$, depends on a distance measure and a kernel function. Let $f_{\hat{r}n}$ be the kernel-based spectral density estimator of the standardized residuals. Proceeding as in Section 2.3.1 leads us to three test statistics given by

$$\begin{aligned}
T_{\mathcal{A}1n} \equiv T_{\mathcal{A}n}(Q^2;k;p_n) &= \frac{nQ^2(f_{\hat{r}n};f_{r0}) - K_{2n}(k)}{\{2K_{4n}(k)\}^{1/2}}, \\
&= \frac{n\sum_{j=1}^{n} k^2(j/p_n)R_{\hat{r}}^2(j) - K_{2n}(k)}{\{2K_{4n}(k)\}^{1/2}}, \\
T_{\mathcal{A}2n} \equiv T_{\mathcal{A}n}(H^2;k;p_n) &= \frac{nH^2(f_{\hat{r}n};f_{r0}) - K_{2n}(k)}{\{2K_{4n}(k)\}^{1/2}}, \\
T_{\mathcal{A}3n} \equiv T_{\mathcal{A}n}(I;k;p_n) &= \frac{nI(f_{\hat{r}n};f_{r0}) - K_{2n}(k)}{\{2K_{4n}(k)\}^{1/2}},
\end{aligned}$$

which are based on the L_2 norm, the Hellinger's distance and the Kullback-Leibler information criterion, respectively. The next theorem establishes the asymptotic distribution of the test statistics under the null hypothesis. We need regularity conditions on $D(\cdot,\cdot)$ and on $\hat{\boldsymbol{\theta}}$. Assumption B states the differentiability conditions on $D(\mathcal{F}_{t-1},\boldsymbol{\theta})$, as a function of the vector parameter $\boldsymbol{\theta}$.

Assumption B:

(1) For each $\boldsymbol{\theta} \in \boldsymbol{\Theta}$, $D(\cdot,\boldsymbol{\theta})$ is a measurable nonnegative function of $\mathcal{F}_{t-1}$; (2)

with probability one, $D(\mathcal{F}_{t-1}, \cdot)$ is twice continuously differentiable with respect to $\boldsymbol{\theta}$ in a neighborhood of $\boldsymbol{\Theta}_0$ of $\boldsymbol{\theta}_0 = \operatorname{plim} \hat{\boldsymbol{\theta}}$, with

$$\lim_{n\to\infty} n^{-1} \sum_{t=1}^{n} E \sup_{\boldsymbol{\theta}\in\boldsymbol{\Theta}_0} ||\frac{\partial}{\partial \boldsymbol{\theta}} D(\mathcal{F}_{t-1}, \boldsymbol{\theta})||^2 < \infty,$$

and

$$\lim_{n\to\infty} n^{-1} \sum_{t=1}^{n} E \sup_{\boldsymbol{\theta}\in\boldsymbol{\Theta}_0} ||\frac{\partial^2}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}'} D(\mathcal{F}_{t-1}, \boldsymbol{\theta})|| < \infty.$$

In Assumption B, the function $D(\mathcal{F}_{t-1}, \boldsymbol{\theta})$ is a given, possibly nonlinear, function such that $D(\cdot, \boldsymbol{\theta})$ is measurable with respect to $\mathcal{F}_{t-1}$ and $D(\mathcal{F}_{t-1}, \cdot)$ is twice differentiable with respect to $\boldsymbol{\theta}$. This hypothesis includes ACD(m) and ACD(m,q) specifications. In the next hypothesis, we suppose that the estimator $\hat{\boldsymbol{\theta}}$ of the true parameter $\boldsymbol{\theta}_0$ is $n^{1/2}$-consistent.

Assumption C: $\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0 = \mathbf{O}_P(n^{-1/2})$.

In Assumption C, any $n^{1/2}$-consistent estimator is allowed. A popular estimator considered in Engle and Russell (1998) is the quasi-maximum likelihood estimator (QMLE), which satisfy Assumption C when the QMLE is based on the exponential distribution. See also Meitz and Teräsvirta (2004). However, the QMLE is often inefficient. An efficient estimator satisfying Assumption C is developed in Drost and Werker (2004). We now state the main result of this section.

Theorem 3 *(a) Assume A(1), B, C,* $E(\epsilon_t^4) < \infty$, $p_n \to \infty$ *and* $p_n/n \to 0$. *Under* $\mathbb{H}_{0,adj}$,

$$T_{\mathcal{A}1n} \to_L N(0,1).$$

(b) Assume the same hypotheses that in (a). Assume furthermore A(2). Under $\mathbb{H}_{0,adj}$, $T_{\mathcal{A}1n} - T_{\mathcal{A}2n} = o_p(1)$ *and* $T_{\mathcal{A}1n} - T_{\mathcal{A}3n} = o_p(1)$.

When the truncated kernel k_{TR} is used, the test statistic $T_{\mathcal{A}1n}$ reduces to

$$T_{\mathcal{A}1n}(Q^2; k_{TR}; p_n) = \frac{n \sum_{j=1}^{p_n} R_{\hat{r}}^2(j) - p_n}{(2p_n)^{1/2}}.$$

Consequently, our approach provides a generalized BP test statistic, when $k = k_{TR}$, under the hypothesis that $p_n \to \infty$ and $p_n/n \to 0$ and Theorem 3 states precise conditions under which the asymptotic distribution of $T_{\mathcal{A}1n}(Q^2; k_{TR}; p_n)$ is normal. More precisely, Theorem 3 gives conditions under which parameter estimation has no impact asymptotically. It suffices to use $n^{1/2}$-consistent estimators for $\boldsymbol{\theta}_0$. Furthermore, we are not restricted to k_{TR} and we have flexibility for the choice of the kernel. Many kernels may have more power than the truncated uniform kernel, by giving more weight to lower orders of lags and less weight to higher orders of lags. The test statistics $T_{\mathcal{A}in}$, for $i \in \{1, 2, 3\}$ are compared for a variety of kernels to BP/LB test statistics in the next section.

2.5 Simulation results

2.5.1 Description of the experiment when testing for ACD effects

We study the finite sample performances of the proposed test statistics for ACD effects $T_{\mathcal{E}in}$, $i \in \{1, 2, 3, 4\}$, and we compare them to BP/LB test statistics. Under $\mathbb{H}_{0,eff}$, $X = \{X_t, t \in \mathbb{Z}\}$ is an iid stochastic process. In order to study the level, we consider the process defined by (2.1) where we set $D_t^0 \equiv 1$. We consider an exponential distribution for the innovation process $\epsilon = \{\epsilon_t, t \in \mathbb{Z}\}$. The power of the test statistics is investigated under the following alternatives:

$$
\begin{aligned}
&\text{ACD(1):} && D_t = 0.8 + 0.2X_{t-1},\\
&\text{ACD(4):} && D_t = 0.8 + 0.1\textstyle\sum_{j=1}^{4}(1 - j/5)X_{t-j},\\
&\text{ACD(12):} && D_t = 0.4 + 0.05\textstyle\sum_{j=1}^{12} X_{t-j},\\
&\text{ACD(1,1):} && D_t = 0.6 + 0.15X_{t-1} + 0.25D_{t-1}.
\end{aligned}
$$

The alternatives are chosen according to their autocorrelation functions and spectral densities. Since the test statistics $T_{\mathcal{E}in}$, $i \in \{1, 2, 3\}$ are function of the estimated spectral density for $\omega \in [-\pi, \pi]$ and $T_{\mathcal{E}4n}$ relies on a spectral density estimator evaluated at frequency $\omega = 0$ only, it seems that the performance of the test statistics will depend in part on the behavior of the spectral density at $\omega = 0$ and the persistence of ACD effects. The ACD(1) alternative has an autocorrelation function which decreases to zero quickly; the shape of the spectral density appears to be regular and it is dominated by low frequencies. The autocorrelation function of the ACD(4) alternative, and more particularly that of the ACD(12) alternative, decrease more slowly. The spectral density of the ACD(12) alternative is dominated by low frequencies with a large spectral density at frequency $\omega = 0$. The ACD(1,1), similarly to the classical ARMA(1,1), appears to be parsimonious in several practical situations. For the chosen alternative, the autocorrelation function decreases rapidly to zero and the spectral density exhibits a regular general behavior, with a moderate spectral peak at $\omega = 0$. We generate time series of length $n = 250, 500, 1000$, which are realistic time series length in several important situations. For example, Bauwens and Giot (2001, p. 54) report sample sizes smaller than 1000 observations for price and volume durations of certain stocks. In the simulations of the ACD processes, the initial values for D_t are set equal to the unconditional mean of X_t. In order to reduce the impact of the initial values, we generate time series of length $2n+1$ and we retain the last n observations. In the level study, $B = 5000$ independent realizations are generated; in the power study the number of realizations is limited to $B = 1000$, for each alternative.

Recall that the BP/LB test statistics for duration clustering are defined by

$$\begin{aligned} \mathrm{BP}(m) &= n\sum_{j=1}^{m} R_X^2(j), \\ \mathrm{LB}(m) &= n^2\sum_{j=1}^{m}(n-j)^{-1}R_X^2(j). \end{aligned}$$

They rely on the choice of m. For fixed m, the test statistics $\mathrm{BP}(m)$ and $\mathrm{LB}(m)$ admit well established asymptotic χ_m^2 distributions under the null hypothesis, because they are based on the raw durations. We considered the test statistics $T_{\mathcal{E}in}$, $i \in \{1,2,3,4\}$, based on the truncated uniform (TR), Bartlett (BAR), Daniell (DAN), Parzen (PAR) and Quadratic Spectral (QS) kernels. We let $m = 6, 10, 16$ when $n = 250$; $m = 7, 11, 20$ when the sample size is $n = 500$, and $m = 7, 12, 24$ when the sample size is $n = 1000$. To facilitate the comparisons, we specify $p_n = m$ for the kernel-based test statistics.

2.5.2 Discussion of the level study (ACD effects)

Table 2.1 reports the simulation results of the level study. Based on $B = 5000$ replications, empirical levels should be in the interval $(4.4\%, 5.6\%)$. Usually, for the considered rates, the test statistics $T_{\mathcal{E}in}$, $i \in \{1,2,3\}$ seem to overreject slightly. Generally, a large value of p_n gives slightly better empirical levels. The test statistic $T_{\mathcal{E}4n}$ has very reasonable levels, but seems to underreject slightly if p_n is large, specially for the truncated uniform and Parzen kernel. The BP/LB test statistics have reasonable levels, particularly LB test statistic, which is a hardly surprising result.

(Insert Table 2.1 here)

2.5.3 Discussion of the power study (ACD effects)

In Tables 2.2-2.5, the power results are given for the alternatives considered. From all these tables, we observe that there is usually little difference in power among Parzen, Daniell and QS kernels. The Bartlett kernel usually gives a more powerful test statistic than these particular kernels. Except for the ACD(12) alternative, the test statistic $T_{\mathcal{E}1n}$ based on k_{TR} is the less powerful. Note that based on the empirical critical values, for a given p_n, $T_{\mathcal{E}1n}$ and BP test statistic have the same adjusted-power, as expected. For most of the alternatives considered, to choose a kernel different from the truncated uniform kernel gives more powerful test statistics. Naturally, the power increases as a function of the sample size. We now discuss specific results under each considered alternative.

Table 2.2 gives the power results under the ACD(1) alternative. The ACD effects are not persistent. As a result, we notice that $T_{\mathcal{E}in}$, $i \in \{1, 2, 3\}$ are more powerful than $T_{\mathcal{E}4n}$. We observe that $T_{\mathcal{E}2n}$ and $T_{\mathcal{E}3n}$ are slightly more powerful than $T_{\mathcal{E}1n}$ in most cases, for this alternative. The test statistic $T_{\mathcal{E}1n}$ is more powerful than BP/LB test statistics for a given order of lag, when the kernel is different from k_{TR}. The power of $T_{\mathcal{E}4n}$ is similar to the power of BP/LB test statistics, for small and moderate p_n and/or a small sample size. Under the ACD(4) alternative reported in Table 2.3, the test statistic $T_{\mathcal{E}4n}$ appears more powerful than $T_{\mathcal{E}in}$, $i \in \{1, 2, 3\}$ and BP/LB test statistics. Accounting for the one-sided alternative seemed to be appropriate in this situation. For $n = 500$ and large p_n, the differences in power are smaller. For $n = 1000$ all the test statistics have comparable power, supporting the asymptotic theory results. The highest power seems for small and moderate p_n for the test statistics $T_{\mathcal{E}in}$, $i \in \{1, 2, 3\}$. However, the test statistic $T_{\mathcal{E}4n}$ has the largest power for small p_n. The ACD(12) alternative described in Table 2.4 generates a spectral peak at frequency zero in the spectral density of the raw duration process; large autocorrelations are present for low and moderate orders of lags. The test statistic $T_{\mathcal{E}4n}$ is more powerful than all the others for sample sizes $n = 250, 500$. For $n = 1000$ the differences in power are smaller, particularly for large p_n. The test statistic $T_{\mathcal{E}1n}$ based on k_{TR} and BP/LB test statistics are more powerful than $T_{\mathcal{E}1n}$ (with $k \neq k_{TR}$), $T_{\mathcal{E}2n}$ and $T_{\mathcal{E}3n}$. The differences are smaller for large p_n when $n = 250$, and the power results are similar when $n = 500, 1000$, specially for large p_n. Finally, Table 2.5 reports the results under the ACD(1,1) alternative. For this alternative, the ACD effects are moderate. Again, the test statistics $T_{\mathcal{E}1n}$ (with $k \neq k_{TR}$), $T_{\mathcal{E}2n}$ and $T_{\mathcal{E}3n}$ are more powerful than BP/LB test statistics for a given p_n. The test $T_{\mathcal{E}4n}$ is the most powerful for small p_n and small n; however, for large p_n, $T_{\mathcal{E}in}$, $i \in \{1, 2, 3\}$ give more powerful test procedures.

(Insert Tables 2.2 - 2.5 here)

2.5.4 Description of the experiment when testing for the adequacy of ACD models

We examine the finite sample performances of the proposed test statistics for the adequacy of ACD models. We compare the new test statistics $T_{\mathcal{A}in}$, $i \in \{1, 2, 3\}$ with the BP/LB test statistics. The proposed test statistics are asymptotically one-sided $N(0, 1)$ under $\mathbb{H}_{0,adj}$. The BP(m)/LB(m) test statistics were intensively used in the literature (e.g., Bauwens and Giot (2001), Engle and Russell (1997, 1998), Tsay (2002)). Their asymptotic distributions are assumed χ^2_m under $\mathbb{H}_{0,adj}$, although no formal analysis, to our knowledge, establishes rigorously the asymptotic distribution of these test statistics. In fact, in view of the results of Li and Yu (2003), the asymptotic distributions of BP/LB test statistics are not χ^2_m under the null hypothesis of adequacy. By comparison, the test statistic $T_{\mathcal{A}n}(Q^2; k_{TR}, p_n)$ provides a generalized BP test statistic and we can investigate empirically if a kernel different from k_{TR} may deliver a higher power in finite samples. We also included BP(m)/LB(m) in our simulations, and we will compare the relative power of these tests to the new test statistics. Since the power of the tests is calculated using the empirical critical values, the power comparison of uniform and non-uniform weighting is valid.

In order to study the level, we considered the process defined by (2.1) where we set $D_t = 0.02 + 0.18X_{t-1} + 0.8D_{t-1}$. We specified an exponential distribution for the innovation process $\epsilon = \{\epsilon_t, t \in \mathbb{Z}\}$. The power of the test statistics has been investigated under the following alternatives:

$$
\begin{array}{ll}
\text{ACD(2,1):} & D_t = 0.2 + 0.3X_{t-1} + 0.4X_{t-2} + 0.1D_{t-1}, \\
\text{ACD(2,2)}^a\text{:} & D_t = 0.2 + 0.1X_{t-1} + 0.3X_{t-2} + 0.1D_{t-1} + 0.3D_{t-2}, \\
\text{ACD(2,2)}^b\text{:} & D_t = 0.2 + 0.1X_{t-1} + 0.2X_{t-2} + 0.1D_{t-2}, \\
\text{ACD(4,4):} & D_t = 0.7 + 0.2X_{t-4} + 0.1D_{t-4}.
\end{array}
$$

As in Section 2.5.1, we generated time series of length $n = 250, 500, 1000$; the test statistics are based on $m \equiv p_n = 6, 10, 16$ when $n = 250$, $m \equiv p_n = 7, 11, 20$ when $n = 500$ and $m \equiv p_n = 7, 12, 24$ when $n = 1000$. We estimated all these data generating processes by an ACD(1,1) model. We calculated the adjusted-power under these alternatives, using the empirical critical values obtained from the level study. When we fit an ACD(1,1) model, the estimated standardized residuals show the remaining dependence in the residuals. For the first three alternatives, the dependence is present in the autocorrelations with low orders of lags. In such situations, it is expected that the kernel-based test statistics will be particularly powerful, since they usually attribute more (less) weight to low (high) orders of lags. The last alternative contains some higher autocorrelations. It is expected that the truncated uniform kernel or the BP/LB test statistics will be powerful, or a kernel-based test statistic with a large p_n. As in Section 2.5.1, we simulated $B = 5000$ replications in the level study, and $B = 1000$ replications for each alternative.

2.5.5 Discussion of the level study (adequacy of ACD models)

Table 2.6 reports the simulation results of the level study for the adjustment test statistics. The test statistics $T_{\mathcal{A}in}$, $i \in \{1, 2, 3\}$ with small p_n and the BP/LB test statistics underreject for $n = 250$. For the kernel-based test statistics, large p_n are generally associated with better empirical levels, since the results are closer of the lower bound of the interval $(4.4\%, 5.6\%)$. The test statistics BP/LB underreject for all p_n. In general, for the considered sample sizes, the kernel-based test statistics have reasonable levels, with most of the empirical levels very close to the interval $(4.4\%, 5.6\%)$. This suggests that large sample sizes are necessary, with appropriate rates for p_n, to obtain empirical critical values close to the asymptotic ones. Interestingly, the generalized BP test statistic, that is $T_{\mathcal{A}1n}$ based on the truncated uniform kernel, has generally better empirical levels than BP/LB test statistics.

(Insert Table 2.6 here)

2.5.6 Discussion of the power study (adequacy of ACD models)

The power results for the adjustment test statistics are given in Tables 2.7-2.10. Similarly to the kernel-based test statistics for ACD effects, there is little difference in power among the considered kernels, if the kernel is chosen different from the truncated uniform kernel. However, for many alternatives, to choose a kernel different from k_{TR} gives more powerful testing procedures than BP/LB test statistics. This is particularly true when the dependence in the residuals happens for autocorrelations with small orders of lags.

More specifically, for the ACD(2,1) alternative and the two ACD(2,2) alternatives under consideration, we find that using a kernel different from k_{TR} gives a more powerful procedure. Usually, $T_{\mathcal{A}1n}$ seems slightly more powerful than $T_{\mathcal{A}2n}$ or $T_{\mathcal{A}3n}$. The power decreases as p_n increases, since a large p_n appears inefficient in the presence of low order dependence.

For the ACD(4,4) alternative, the residuals contain higher order dependence. As a result, to choose a low p_n is inefficient for the kernel-based test statistics, since the test statistics $T_{\mathcal{A}1n}$ based on k_{TR} or the BP/LB test statistics with $p_n = 6$ $(n = 250)$ or $p_n = 7$ $(n = 500, 1000)$, are more powerful than $T_{\mathcal{A}in}$, $i \in \{1, 2, 3\}$ with $k \neq k_{TR}$. However, for a moderate or high value of p_n, we obtain more powerful procedures with the kernel-based test statistics based on a non-uniform weighting scheme. This happens because for a given order of lag, say j_0, a larger p_n attributes a larger weight to the autocorrelation of lag j_0 (more formally, for a fixed j_0, $k(j_0/p_n) \to 1$ as $n \to \infty$). More simulation results can be found in the working paper Duchesne and Pacurar (2003).

(Insert Tables 2.7 - 2.10 here)

2.6 Application with IBM data

In this section we analyze the IBM transaction data taken from the TORQ (Trades, Orders, Reports, and Quotes) data set compiled by Hasbrouck (1991) and the New York Stock Exchange (NYSE). The same database was used by Engle and Russell (1998) to implement the ACD models. The NYSE is the world's largest equities market with a \$17.8 trillion global market capitalization as of September, 2004 (Source: NYSE Fact Book). The market combines the features of a price-driven market (presence of a market maker) with those of an order-driven market (existence of an order book), which makes it an explicitly hybrid mechanism. Specifically, each stock is allocated to a single market maker, named specialist, who must ensure an orderly market in that stock and therefore commit his own capital while taking position against the trend of the market until stability is achieved. However, the specialist must also monitor the order book for the stock in which limit orders are transmitted electronically or via a floor trader (see Hasbrouck *et al.* (1993) for a description of the operations of the NYSE). The NYSE opens with a call auction implemented to find the opening price and then trading occurs continuously from 9h30 to 16h.

In our study, we focus on two duration data sets: a large one consisting of trade durations (i.e. time intervals between successive trades) and a small one consisting of volume durations (i.e. the times between trades until a given cumulated volume is traded on the market). Trade durations are indicators for the trading activity and they have received much attention in the financial literature on market microstructure. For example, several studies were concerned with how the timing of trades affects market behavior and the formation of prices. The asymmetric information models, developed in more recent research on market microstructure assume that trades convey information. If some traders are better informed than others, it seems plausible in such circumstances that their trades could reveal some information. Glosten and Milgrom (1985), Easley and O'Hara (1992), Diamond and Verrecchia (1987) and Admati and Pfleiderer (1988), among others, emphasized this notion of time as signal and the role of intertrade durations for explaining how the information is processed in financial markets. O'Hara (1995) provides an excellent review of the market microstructure theory. Volume durations were introduced by Gouriéroux, Jasiak, and Le Fol (1999) as reasonable measures of liquidity that account simultaneously for the time and volume dimension of the trading process. According to the conventional definition of liquidity (Demsetz, 1968; Black, 1971; Glosten and Harris, 1988), an asset is liquid if it can be traded as fast as possible, in large quantities, and with no significant impact on the price. Consequently, while neglecting the price impact of volumes, volume durations may still be interpreted as the (time) cost of liquidity. For a more detailed analysis of different types of volume durations, see Hautsch (2001).

Our sample period runs over three months, from November 1, 1990 through January 31, 1991. The initial sample contains 60328 transactions for a total of 63 trading days. For each transaction, the information recorded contains the calendar date, a time stamp measured in seconds after midnight at which the transaction took place, the volume of shares traded, the bid and ask price at the time of trade and finally the transaction price. We follow the lines of the work of Engle and Russell (1998) and Engle (2000) for cleaning the data. The interdaily durations as well as transactions occurring outside regular trading hours were removed. Moreover, transactions on Thanksgiving Friday and the day before Christmas and New Year were deleted. Trades occurring simultaneously (thus leading to zero durations) were also discarded. This uses the microstructure argument that zero durations correspond to 'split-transactions' (i.e. one big order which is matched against several smaller opposite orders). Veredas, Rodriguez-Poo, and Espasa (2001) propose another way for dealing with zero durations but for simplicity, we proceeded as in Engle and Russell (1998) and Engle (2000). The sample thus reduced to 52146 observations resulting in 52145 trade durations. We compute volume durations based on an aggregated volume of 90000 shares. This leads to a reduced sample of 1025 volume durations which is comparable to the largest sample size used in our simulations.

Table 2.11 reports some descriptive statistics for each type of duration process. During the period analyzed, the IBM stock was traded on average every 26.64 seconds with a standard deviation of 36.62. The minimum trade duration is 1 second and the maximum duration is 561 seconds. It took on average 22 minutes for trading 90 000 IBM shares. Consistent with what has been extensively documented in the literature, trade durations are overdispersed while volume durations are underdispersed.

As evidenced in previous empirical work, both types of durations exhibit a strong intra-day seasonality, being shortest near the open and prior to the close of the market. Therefore, as in Engle (2000), prior to the estimation of any ACD model we compute the diurnally adjusted durations by regressing the durations on the time of the day using piecewise linear splines and then taking the ratios of durations to fitted values. This procedure and other approaches are discussed in Bauwens and Giot (2001). See also Tsay (2002) for a slightly different approach using quadratic functions and indicator variables. The coefficient of variation of the seasonally adjusted trade (volume) durations series is less pronounced than in the raw durations series but it is still greater (smaller) than 1. As we can see from Table 2.12, the test statistics for ACD effects based on adjusted trade durations are generally reduced but they are all highly significant, revealing persistence in trade durations. For volume durations (see Table 2.13), the test statistics for ACD effects actually increased after deseasonalization suggesting that a more refined deseasonalization technique could be worth investigating. The test statistics in Tables 2.12 and 2.13 indicate a stronger persistence for trade durations than for volume durations. Large positive autocorrelations, overdispersion (underdispersion) and the fact that large (small) durations tend to be followed by large (small) durations (which is similar to a GARCH mechanism, see Bollerslev, Engle and Nelson (1994)), seem to be the main features documented in the literature for trade (volume) durations that justify the introduction of ACD models.

Since the test statistics from Tables 2.12 and 2.13 suggest the presence of ACD effects at any reasonable significance level, we now proceed to the estimation of ACD models on IBM data. We consider ACD(1,1) and ACD(2,2) models with several specifications for the distribution of the innovation ϵ_t in (2.1). We specify for the distribution $p_\epsilon(\cdot)$ the exponential, Weibull and generalized gamma distribution, giving us the EACD, WACD and GACD models, respectively. We wrote the log-likelihood for all these cases and we performed the optimization via the Nelder-Mead simplex method, using MATLAB software. We applied our test procedures $T_{\mathcal{A}1n}$ with the Daniell (k_{DAN}), Bartlett (k_{BAR}) and truncated uniform (k_{TR}) kernels.

Tables 2.14, 2.15 and 2.16 report the test statistics with the corresponding p-values for trade durations. The ACD(1,1) seems inappropriate since all the p-values are highly significant. The ACD(2,2) provides an improvement over the ACD(1,1) model, since the p-values of the test statistics are generally higher. Since the truncated uniform kernel attributes equal weighting for each considered lag, it seems plausible that high residual autocorrelations are still present in the EACD(2,2) and WACD(2,2) models, since for $p_n = 16$ the test statistic $T_{\mathcal{A}1n}(k_{TR})$ rejects the null hypothesis of adequacy, and for $p_n = 24$ the same can be said for $T_{\mathcal{A}1n}(k_{BAR})$ and $T_{\mathcal{A}1n}(k_{DAN})$. It seems that the GACD(2,2) gives the best fit for these data, since the p-values of the test statistics $T_{\mathcal{A}1n}(k_{BAR})$ and $T_{\mathcal{A}1n}(k_{DAN})$ are larger than 5% or very close to it. The test statistic $T_{\mathcal{A}1n}(k_{TR})$ seems to suggest that some large autocorrelations are still present, since $T_{\mathcal{A}1n}(k_{TR})$ with $p_n = 16$ is still significant. However, with more that 50000 observations, that test statistic is very close to the 1% significance level. Furthermore, the test statistics with Bartlett and Daniell kernels, which have been shown to be more powerful than $T_{\mathcal{A}1n}(k_{TR})$ in many situations (see the simulations of the Section 2.5), seem to indicate the adequacy of the GACD(2,2) model. Overall, we conclude that the GACD(2,2) provides a rather satisfactory adjustment for trade durations over the GACD(1,1) model since with this sample size, the test statistics from Table 2.16 do not clearly indicate an inadequate model.

Tables 2.17, 2.18 and 2.19 report the test statistics with the corresponding p-values for volume durations. Like for the trade durations, the ACD(1,1) models are largely rejected, yet some of the p-values, especially for a large p_n are superior to 5%. The adjustments of the EACD(2,2) and WACD(2,2) models capture adequately the duration dependence since all the p-values of the test statistics are highly insignificant. Consequently, these models seem to provide the most parsimonious models for our sample of volume durations. Interestingly, the overall performance of the GACD(2,2) model for volume durations is not satisfying since rejection of the null hypothesis of adequacy is indicated by $T_{\mathcal{A}1n}(k_{BAR})$ and $T_{\mathcal{A}1n}(k_{DAN})$ for $p_n = 7$. Given the argument pointed out in the literature that the observed hump-shaped density of volume durations requires a more flexible innovation distribution, we would have expected a better performance of the GACD model. We have also estimated ACD models based on a Burr distribution but the results were not improved.

2.7 Conclusion

In this paper, we have proposed two classes of test statistics for duration clustering and a new class of diagnostic test statistics for ACD models, using a spectral approach. When testing for ACD effects, the test statistics in the first class are based on a distance measure and a kernel-based spectral density estimator of the raw durations. On the other side, the test statistics in the second class exploit the one-sided nature of duration clustering. Asymptotic arguments suggest that the test statistics in the first class should be more powerful asymptotically, but to exploit the one-sided nature of the alternative hypothesis may be particularly powerful in small samples. A small sample size could occur when the market participants are interested in volume durations, among others. When testing for adequacy, we proposed a class of test statistics using a kernel-based spectral density estimator of the estimated standardized duration residuals. We established the asymptotic distributions of the test statistics for duration clustering and for the adequacy of ACD models, which are normal under their respective null hypothesis. We also discussed when the tests in a given class are asymptotically equivalent under the null hypothesis.

Empirical experiments have been conducted to evaluate the proposed procedures. We found that all the proposed test statistics had reasonable levels. Typically, the weighting scheme of the kernel-based test statistics attributes more (less) weight to low (high) orders of lags. We found that in many circumstances such weighting contributes to powerful procedures. In many situations, the proposed test statistics were more powerful than BP/LB test statistics. When ACD effects were particularly persistent, the test statistics in the second class were usually more powerful. Similarly, when testing for the adequacy of ACD models, the proposed test statistics had reasonable levels. Interestingly, our generalized BP test statistic appeared to have better empirical levels than BP/LB test statistics, at least in our experiments. Concerning the power, usually a kernel different from the truncated uniform kernel led to more powerful procedures, since the kernel-based test statistics (with $k \neq k_{TR}$), were more powerful than BP/LB test statistics, for the chosen models.

We illustrated our test procedures on the IBM dataset considered in Engle and Russell (1998). We found that the GACD(2,2) provided a rather satisfactory adjustment for trade durations, since with a sample size of more that 50000 observations, the test statistics based on the truncated uniform, Bartlett and Daniell kernels, did not clearly indicate an inadequate model. For volume durations, EACD(2,2) and WACD(2,2) provided parsimonious models.

2.8 References

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2.9 Appendix

Proof of the theorem 1

Let $\tilde{T}_{\mathcal{E}1n} = \{n\sum_{j=1}^{n-1} k^2(j/p_n)\tilde{R}_X^2(j) - K_{2n}(k)\}/\{2K_{4n}(k)\}^{1/2}$, where $\tilde{R}_X(j) = \tilde{C}_X(j)/\tilde{C}_X(0)$ and $\tilde{C}_X(j) = n^{-1}\sum_{t=j+1}^{n}(X_t - \mu_X)(X_{t-j} - \mu_X)$, $\mu_X = EX_t$. According to the Theorem A.1 of Hong (1996), we deduce that $\tilde{T}_{\mathcal{E}1n} \to_L N(0,1)$ under the null hypothesis. Let $k_{jn} = k(j/p_n)$. To establish the Theorem 1 it suffices to show that $\sum_{j=1}^{n-1} k_{jn}^2\{\tilde{R}_X^2(j) - R_X^2(j)\} = o_P(p_n^{1/2}/n)$. We can write $\sum_{j=1}^{n-1} k_{jn}^2\{\tilde{R}_X^2(j) - R_X^2(j)\} = \tilde{C}_X^{-2}(0)\sum_{j=1}^{n-1} k_{jn}^2\{\tilde{C}_X^2(j) - C_X^2(j)\} + \{\tilde{C}_X^{-2}(0) - C_X^{-2}(0)\}\sum_{j=1}^{n-1} k_{jn}^2 C_X^2(j)$. Note that $\sum_{j=1}^{n-1} k_{jn}^2 C_X^2(j) = O_p(p_n/n)$ (see Hong(1996, p. 854)). Since $\tilde{C}_X(0) - C_X(0) = (\bar{X} - \mu_X)^2$, we deduce that $\tilde{C}_X(0) - C_X(0) = O_P(n^{-1})$. This shows that $(\tilde{C}_X^{-2}(0) - C_X^{-2}(0))\sum_{j=1}^{n-1} k_{jn}^2 C_X^2(j) = O_P(p_n/n^2)$. Write

$$\sum_{j=1}^{n-1} k_{jn}^2\{\tilde{C}_X^2(j) - C_X^2(j)\} = \sum_{j=1}^{n-1} k_{jn}^2\{\tilde{C}_X(j) - C_X(j)\}^2 + 2\sum_{j=1}^{n-1} k_{jn}^2\tilde{C}_X(j)\{\tilde{C}_X(j) - C_X(j)\}.$$

We decompose $C_X(j) - \tilde{C}_X(j) = (\bar{X} - \mu_X)\{-\bar{X}^+ - \bar{X}^- + \frac{n-j}{n}(\bar{X} + \mu_X)\}$, where $\bar{X}^+ = n^{-1}\sum_{t=j+1}^{n} X_t$ and $\bar{X}^- = n^{-1}\sum_{t=j+1}^{n} X_{t-j}$. We deduce that $C_X(j) - \tilde{C}_X(j) = O_P(n^{-1})$, uniformly in j, and $\sum_{j=1}^{n-1} k_{jn}^2\{C_X(j) - \tilde{C}_X(j)\}^2 = O_P(p_n/n^2)$. Using Cauchy-Schwartz inequality, we can show that

$$\sum_{j=1}^{n-1} k_{jn}^2\tilde{C}_X(j)\{C_X(j) - \tilde{C}_X(j)\} = O_P(p_n/n^{3/2}).$$

This concludes the proof for $T_{\mathcal{E}1n}$. We now prove the result for $T_{\mathcal{E}2n}$ and $T_{\mathcal{E}3n}$. We now introduce $\tilde{f}_X(\omega) = \frac{1}{2\pi}\sum_{h=-n+1}^{n-1} k_{jn}\tilde{R}_X(h)e^{-ih\omega}$. We have that

$$||\hat{f}_X - \tilde{f}_X||_\infty \equiv \sup_{\omega\in[-\pi,\pi]} |\hat{f}_X(\omega) - \tilde{f}_X(\omega)| \leq \frac{1}{2\pi}\sum_{j=-n+1}^{n-1} |k_{jn}||R_X(j) - \tilde{R}_X(j)| = O_P(p_n/n),$$

and also $||\tilde{f}_X - f_{X0}||_\infty = O_p(p_n/n^{1/2})$ (see Hong (1996, p. 860)). Consequently, $||\hat{f}_X - f_{X0}||_\infty = O_P(p_n/n^{1/2})$, which is the key step. Following Hong (1996, Theorem 3), under the assumption $p_n^3/n \to 0$, it follows that $T_{\mathcal{E}1n} - T_{\mathcal{E}2n} = o_P(1)$ and $T_{\mathcal{E}1n} - T_{\mathcal{E}3n} = o_P(1)$ and the result follows.

Proof of the theorem 2

We prove the result when $p_n \to \infty$. The proof for fixed p_n is similar and simpler. Let $Y_t = X_t/\mu_X$. Note that we can write $\tilde{R}_X(j) = \tilde{C}_Y(j)/\tilde{C}_Y(0)$, where $\tilde{C}_Y(j) = n^{-1}\sum_{t=j+1}^{n}(Y_t-1)(Y_{t-j}-1)$. Using the Lemma A.2 of Hong (1997), we deduce that $K_{2n}^{-1/2}(k)n^{1/2}\sum_{j=1}^{n-1} k_{jn}\tilde{C}_Y(j)/E\{\tilde{C}_Y(0)\} \to_L N(0,1)$ under the null hypothesis. We have to show that $K_{2n}^{-1/2}(k)n^{1/2}\sum_{j=1}^{n-1} k_{jn}\{\tilde{C}_X(j) - C_X(j)\} = o_P(1)$, that is $\sum_{j=1}^{n-1} k_{jn}\{\tilde{C}_X(j) - C_X(j)\} = o_P(p_n^{1/2}/n^{1/2})$. We already established that $\tilde{C}_X(j) - C_X(j) = O_P(n^{-1})$ uniformly in j. This shows that

$$\sum_{j=1}^{n-1} k_{jn}\{\tilde{C}_X(j) - C_X(j)\} = O_P(p_n/n) = o_P(p_n^{1/2}/n^{1/2}),$$

since $p_n/n \to 0$. This concludes the proof.

Proof of the theorem 3

We begin to establish the asymptotic distribution of $T_{\mathcal{A}1n}$. Let $\tilde{T}_{\mathcal{A}1n} = \{n\sum_{j=1}^{n-1} k_{jn}^2 R_Z^2(j) - K_{2n}(k)\}/\{2K_{4n}(k)\}^{1/2}$, where $R_Z(j) = C_Z(j)/C_Z(0)$, $C_Z(j) = n^{-1}\sum_{t=j+1}^{n} Z_t Z_{t-j}$, $Z_t = r_t - 1$. Under the null hypothesis, we have that $r_t = \epsilon_t$, a.s., and $E(Z_t) = E(\epsilon_t - 1) = 0$. Since the process $\{r_t, t \in \mathbb{Z}\}$ is iid under the null, it follows using the same kind of arguments that in Theorem 1 that $T_{\mathcal{A}1n} \to_L N(0,1)$ under the null hypothesis. We have to establish that $\tilde{T}_{\mathcal{A}1n} - T_{\mathcal{A}1n} = o_P(1)$. In order to show that $\sum_{j=1}^{n-1} k_{jn}^2\{R_{\hat{r}}^2(j) - R_Z^2(j)\} = o_P(p_n^{1/2}/n)$, we will show that

$$\sum_{j=1}^{n-1} k_{jn}^2\{C_{\hat{Z}}(j) - C_Z(j)\}^2 = O_P(p_n/n^2 + n^{-1}), \tag{2.12}$$

$$\sum_{j=1}^{n-1} k_{jn}^2 C_Z(j)\{C_{\hat{Z}}(j) - C_Z(j)\} = O_P(p_n/n^{3/2} + n^{-1}). \qquad (2.13)$$

We first show (2.12). A Taylor's expansion of the function $D_t^{-1}(\hat{\boldsymbol{\theta}})$ gives

$$D_t^{-1}(\hat{\boldsymbol{\theta}}) = D_t^{-1}(\boldsymbol{\theta}_0)+\frac{\partial}{\partial \boldsymbol{\theta}}D_t^{-1}(\boldsymbol{\theta}_0)(\hat{\boldsymbol{\theta}}-\boldsymbol{\theta}_0)+\frac{1}{2}(\hat{\boldsymbol{\theta}}-\boldsymbol{\theta}_0)'\frac{\partial^2}{\partial \boldsymbol{\theta}\partial \boldsymbol{\theta}'}D_t^{-1}(\bar{\boldsymbol{\theta}})(\hat{\boldsymbol{\theta}}-\boldsymbol{\theta}_0), \quad (2.14)$$

where $\bar{\boldsymbol{\theta}}$ lies between $\hat{\boldsymbol{\theta}}$ and $\boldsymbol{\theta}_0$. We decompose the difference $C_{\hat{Z}}(j) - C_Z(j)$ as

$$\begin{aligned} C_{\hat{Z}}(j) - C_Z(j) &= n^{-1}\sum_{t=j+1}^{n}(\hat{Z}_t - Z_t)(\hat{Z}_{t-j} - Z_{t-j}) + n^{-1}\sum_{t=j+1}^{n} Z_t(\hat{Z}_{t-j} - Z_{t-j}) \\ &\quad + n^{-1}\sum_{t=j+1}^{n}(\hat{Z}_t - Z_t)Z_{t-j}, \\ &= \hat{A}_{1j} + \hat{A}_{2j} + \hat{A}_{3j}. \end{aligned}$$

Noting that $(\hat{A}_{1j}+\hat{A}_{2j}+\hat{A}_{3j})^2 \leq 4\{\hat{A}_{1j}^2+\hat{A}_{2j}^2+\hat{A}_{3j}^2\}$, we can write $\sum_{j=1}^{n-1} k_{jn}^2\{C_{\hat{Z}}(j)-C_Z(j)\}^2 \leq 4(\hat{B}_{1n}+\hat{B}_{2n}+\hat{B}_{3n})$, where $\hat{B}_{1n} = \sum_{j=1}^{n-1} k_{jn}^2 \hat{A}_{1j}^2$, $\hat{B}_{2n} = \sum_{j=1}^{n-1} k_{jn}^2 \hat{A}_{2j}^2$ and $\hat{B}_{3n} = \sum_{j=1}^{n-1} k_{jn}^2 \hat{A}_{3j}^2$. We study individually each term. We have that $\hat{Z}_t - Z_t = X_t\{D_t^{-1}(\hat{\boldsymbol{\theta}}) - D_t^{-1}(\boldsymbol{\theta}_0)\}$. Replacing in $\hat{A}_{1j}$, using Assumption C, we can show easily that $\sum_{j=1}^{n-1} k_{jn}^2 \hat{A}_{1j}^2 = O_P(p_n/n^2)$. We now turn to $\hat{B}_{2n}$. We can decompose $\hat{A}_{2j} = \hat{A}_{21j} + \hat{A}_{22j}$, $\hat{A}_{21j} = \hat{a}_{21j}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0)$, $\hat{A}_{22j} = (\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0)'\hat{a}_{22j}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0)$, where $\hat{a}_{21j} = n^{-1}\sum_{t=j+1}^{n} Z_t X_{t-j}\frac{\partial}{\partial \boldsymbol{\theta}}D_{t-j}^{-1}(\boldsymbol{\theta}_0)$, $\hat{a}_{22j} = \frac{1}{2}n^{-1}\sum_{t=j+1}^{n} Z_t X_{t-j}\frac{\partial^2}{\partial \boldsymbol{\theta}\partial \boldsymbol{\theta}'}D_{t-j}^{-1}(\bar{\boldsymbol{\theta}})$. It is easy to show that $\hat{a}_{21j} = O_P(n^{-1/2})$ since $E(Z_t|\mathcal{F}_{t-1}) = E(Z_t) = 0$. Using $\hat{A}_{2j}^2 \leq 2(\hat{A}_{21j}^2 + \hat{A}_{22j}^2)$, we deduce that $\sum_{j=1}^{n-1} k_{jn}^2 \hat{A}_{2j}^2 = O_P(p_n/n^2)$. We now study the term $\hat{B}_{3n}$. We write $\hat{A}_{3j} = \hat{A}_{31j} + \hat{A}_{32j}$, $\hat{A}_{31j} = \hat{a}_{31j}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0)$, $\hat{A}_{32j} = (\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0)'\hat{a}_{32j}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0)$, where $\hat{a}_{31j} = n^{-1}\sum_{t=j+1}^{n} X_t Z_{t-j}\frac{\partial}{\partial \boldsymbol{\theta}}D_t^{-1}(\boldsymbol{\theta}_0)$, $\hat{a}_{32j} = \frac{1}{2}n^{-1}\sum_{t=j+1}^{n} Z_{t-j}X_t\frac{\partial^2}{\partial \boldsymbol{\theta}\partial \boldsymbol{\theta}'}D_t^{-1}(\bar{\boldsymbol{\theta}})$. We now study separately $\hat{a}_{31j}$ and $\hat{a}_{32j}$. Using $\hat{A}_{3j}^2 \leq 2(\hat{A}_{31j}^2 + \hat{A}_{32j}^2)$, we have that

$$\hat{B}_{3n} \leq 2\left(\sum_{j=1}^{n-1} k_{jn}^2 \hat{A}_{31j}^2 + \sum_{j=1}^{n-1} k_{jn}^2 \hat{A}_{32j}^2\right).$$

We show easily that $\sum_{j=1}^{n-1} k_{jn}^2 \hat{A}_{32j}^2 = O_P(p_n/n^2)$. Since $|\hat{A}_{31j}^2| \leq ||\hat{a}_{31j}||^2 ||\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0||^2$, we have that

$$\sum_{j=1}^{n-1} k_{jn}^2 \hat{A}_{31j}^2 \leq ||\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0||^2 \sum_{j=1}^{n-1} k_{jn}^2 ||\hat{a}_{31j}||^2.$$

We write $\hat{a}_{31j} = \hat{a}_{31j} - E(\hat{a}_{31j}) + E(\hat{a}_{31j})$. Note that we can interpret $E(\hat{a}_{31j})$ as a cross-correlation, since $E(\hat{a}_{31j}) = \operatorname{cov}(X_t \frac{\partial}{\partial \boldsymbol{\theta}} D_t^{-1}(\boldsymbol{\theta}_0), Z_{t-j})$. Consequently, $\sum_{j=1}^{n-1} k_{jn}^2 ||E(\hat{a}_{31j})||^2 = O(1)$. We show easily that $\sum_{j=1}^{n-1} k_{jn}^2 ||\hat{a}_{31j} - E(\hat{a}_{31j})||^2 = O_P(p_n/n)$. This shows the first part (2.12). We now work the second part (2.13). We have that

$$\begin{aligned}
\sum_{j=1}^{n-1} k_{jn}^2 C_Z(j)\{C_{\hat{Z}}(j) - C_Z(j)\} &= \sum_{j=1}^{n-1} k_{jn}^2 C_Z(j)\{\hat{A}_{1j} + \hat{A}_{2j} + \hat{A}_{3j}\}, \\
&= \hat{D}_{1n} + \hat{D}_{2n} + \hat{D}_{3n}.
\end{aligned}$$

We begin with $\hat{D}_{1n}$. Using Cauchy-Schwartz inequality,

$$\left| \sum_{j=1}^{n-1} k_{jn}^2 C_Z(j) \hat{A}_{1j} \right| \leq \left\{ \sum_{j=1}^{n-1} k_{jn}^2 C_Z^2(j) \right\}^{1/2} \left\{ \sum_{j=1}^{n-1} k_{jn}^2 \hat{A}_{1j}^2 \right\}^{1/2}.$$

Since $\sum_{j=1}^{n-1} k_{jn}^2 C_Z^2(j) = O_P(p_n/n)$ and $\sum_{j=1}^{n-1} k_{jn}^2 \hat{A}_{1j}^2 = O_P(p_n/n^2)$, this shows that $\sum_{j=1}^{n-1} k_{jn}^2 C_Z(j) \hat{A}_{1j} = O_P(p_n/n^{3/2})$. Similarly, $\sum_{j=1}^{n-1} k_{jn}^2 C_Z(j) \hat{A}_{2j} = O_P(p_n/n^{3/2})$. For the term involving $\hat{A}_{3j}$, we have that

$$\sum_{j=1}^{n-1} k_{jn}^2 C_Z(j) \hat{A}_{3j} = \sum_{j=1}^{n-1} k_{jn}^2 C_Z(j) \hat{A}_{31j} + \sum_{j=1}^{n-1} k_{jn}^2 C_Z(j) \hat{A}_{32j}.$$

We show easily that $\sum_{j=1}^{n-1} k_{jn}^2 C_Z(j) \hat{A}_{32j} = O_P(p_n/n^{3/2})$. Since

$$\sum_{j=1}^{n-1} k_{jn}^2 C_Z(j) \hat{A}_{31j} = \sum_{j=1}^{n-1} k_{jn}^2 C_Z(j) \hat{a}_{31j} (\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0),$$

we show that $\sum_{j=1}^{n-1} k_{jn}^2 C_Z(j) \hat{A}_{32j} = O_P(p_n/n^{3/2} + n^{-1})$. This shows (2.13).

We now prove the result for $T_{\mathcal{A}2n}$ and $T_{\mathcal{A}3n}$. We recall that $\hat{f}_Z(\omega) =$

$\frac{1}{2\pi}\sum_{h=-n+1}^{n-1} k_{jn} R_Z(h) e^{-ih\omega}$ and we introduce $\tilde{f}_Z(\omega) = \frac{1}{2\pi}\sum_{h=-n+1}^{n-1} k_{jn} \tilde{R}_Z(h) e^{-ih\omega}$. We have that

$$\begin{aligned} ||\hat{f}_Z - \tilde{f}_Z||_\infty &\equiv \sup_{\omega\in[-\pi,\pi]} |\hat{f}_Z(\omega) - \tilde{f}_Z(\omega)|, \\ &\leq \frac{1}{2\pi}\sum_{j=-n+1}^{n-1} |k_{jn}||R_Z(j) - \tilde{R}_Z(j)|, \\ &= O_P(p_n/n^{1/2}). \end{aligned}$$

As in the proof of Theorem 1, we have that $||\tilde{f}_Z - f_{Z0}||_\infty = O_p(p_n/n^{1/2})$. Consequently, $||\hat{f}_Z - f_{Z0}||_\infty = O_P(p_n/n^{1/2})$, as in the proof of Theorem 1. Consequently, under the assumption $p_n^3/n \to 0$, it follows that $T_{\mathcal{A}1n} - T_{\mathcal{A}2n} = o_P(1)$ and $T_{\mathcal{A}1n} - T_{\mathcal{A}3n} = o_P(1)$ and the result follows.

Table 2.1: Empirical levels at the 5% level for tests of ACD effects

$n = 250$					$n = 500$					$n = 1000$				
$p_n = 6$	$T_{\mathcal{E}1n}$	$T_{\mathcal{E}2n}$	$T_{\mathcal{E}3n}$	$T_{\mathcal{E}4n}$	$p_n = 7$	$T_{\mathcal{E}1n}$	$T_{\mathcal{E}2n}$	$T_{\mathcal{E}3n}$	$T_{\mathcal{E}4n}$	$p_n = 7$	$T_{\mathcal{E}1n}$	$T_{\mathcal{E}2n}$	$T_{\mathcal{E}3n}$	$T_{\mathcal{E}4n}$
TR	0.0618			0.0430	TR	0.0670			0.0476	TR	0.0666			0.0498
BAR	0.0616	0.0636	0.0628	0.0454	BAR	0.0634	0.0640	0.0638	0.0466	BAR	0.0708	0.0708	0.0702	0.0478
DAN	0.0616	0.0632	0.0648	0.0492	DAN	0.0610	0.0630	0.0638	0.0516	DAN	0.0712	0.0712	0.0742	0.0490
PAR	0.0656	0.0658	0.0664	0.0432	PAR	0.0626	0.0626	0.0616	0.0480	PAR	0.0694	0.0704	0.0700	0.0478
QS	0.0638	0.0652	0.0652	0.0448	QS	0.0624	0.0630	0.0632	0.0478	QS	0.0708	0.0710	0.0718	0.0482
LB(p_n)	0.0388				LB(p_n)	0.0456				LB(p_n)	0.0478			
BP(p_n)	0.0414				BP(p_n)	0.0468				BP(p_n)	0.0480			
$p_n = 10$	$T_{\mathcal{E}1n}$	$T_{\mathcal{E}2n}$	$T_{\mathcal{E}3n}$	$T_{\mathcal{E}4n}$	$p_n = 11$	$T_{\mathcal{E}1n}$	$T_{\mathcal{E}2n}$	$T_{\mathcal{E}3n}$	$T_{\mathcal{E}4n}$	$p_n = 12$	$T_{\mathcal{E}1n}$	$T_{\mathcal{E}2n}$	$T_{\mathcal{E}3n}$	$T_{\mathcal{E}4n}$
TR	0.0556			0.0450	TR	0.0656			0.0474	TR	0.0668			0.0480
BAR	0.0632	0.0644	0.0656	0.0442	BAR	0.0632	0.0612	0.0622	0.0484	BAR	0.0696	0.0700	0.0710	0.0498
DAN	0.0628	0.0656	0.0720	0.0478	DAN	0.0640	0.0644	0.0672	0.0524	DAN	0.0668	0.0696	0.0726	0.0508
PAR	0.0628	0.0636	0.0692	0.0436	PAR	0.0642	0.0642	0.0664	0.0498	PAR	0.0672	0.0676	0.0714	0.0514
QS	0.0638	0.0664	0.0708	0.0444	QS	0.0642	0.0648	0.0662	0.0482	QS	0.0670	0.0696	0.0712	0.0498
LB(p_n)	0.0334				LB(p_n)	0.0442				LB(p_n)	0.0456			
BP(p_n)	0.0378				BP(p_n)	0.0470				BP(p_n)	0.0472			
$p_n = 16$	$T_{\mathcal{E}1n}$	$T_{\mathcal{E}2n}$	$T_{\mathcal{E}3n}$	$T_{\mathcal{E}4n}$	$p_n = 20$	$T_{\mathcal{E}1n}$	$T_{\mathcal{E}2n}$	$T_{\mathcal{E}3n}$	$T_{\mathcal{E}4n}$	$p_n = 24$	$T_{\mathcal{E}1n}$	$T_{\mathcal{E}2n}$	$T_{\mathcal{E}3n}$	$T_{\mathcal{E}4n}$
TR	0.0574			0.0410	TR	0.0638			0.0410	TR	0.0626			0.0460
BAR	0.0618	0.0588	0.0626	0.0432	BAR	0.0610	0.0606	0.0626	0.0476	BAR	0.0690	0.0682	0.0688	0.0504
DAN	0.0598	0.0592	0.0680	0.0468	DAN	0.0648	0.0630	0.0682	0.0494	DAN	0.0674	0.0690	0.0744	0.0510
PAR	0.0590	0.0552	0.0624	0.0426	PAR	0.0632	0.0594	0.0646	0.0458	PAR	0.0690	0.0664	0.0712	0.0506
QS	0.0606	0.0572	0.0658	0.0438	QS	0.0630	0.0620	0.0674	0.0468	QS	0.0674	0.0676	0.0726	0.0506
LB(p_n)	0.0278				LB(p_n)	0.0344				LB(p_n)	0.0428			
BP(p_n)	0.0340				BP(p_n)	0.0422				BP(p_n)	0.0456			

1) DGP:X$_t = \epsilon_t, \epsilon_t \sim EXP(1)$ 2) 5000 iterations.

Table 2.2: Level-adjusted powers against ACD(1) at 5% level for tests of ACD effects

	$n = 250$					$n = 500$					$n = 1000$			
$p_n = 6$	$T_{\mathcal{E}1n}$	$T_{\mathcal{E}2n}$	$T_{\mathcal{E}3n}$	$T_{\mathcal{E}4n}$	$p_n = 7$	$T_{\mathcal{E}1n}$	$T_{\mathcal{E}2n}$	$T_{\mathcal{E}3n}$	$T_{\mathcal{E}4n}$	$p_n = 7$	$T_{\mathcal{E}1n}$	$T_{\mathcal{E}2n}$	$T_{\mathcal{E}3n}$	$T_{\mathcal{E}4n}$
TR	0.547			0.405	TR	0.840			0.552	TR	0.987			0.766
BAR	0.717	0.724	0.732	0.632	BAR	0.935	0.936	0.939	0.844	BAR	1.000	1.000	1.000	0.972
DAN	0.697	0.704	0.719	0.575	DAN	0.928	0.934	0.935	0.784	DAN	0.998	0.998	0.998	0.946
PAR	0.684	0.683	0.690	0.582	PAR	0.925	0.921	0.927	0.769	PAR	0.998	0.997	0.999	0.950
QS	0.694	0.699	0.718	0.609	QS	0.928	0.931	0.935	0.811	QS	0.998	0.998	0.999	0.958
LB(p_n)	0.549				LB(p_n)	0.841				LB(p_n)	0.987			
BP(p_n)	0.547				BP(p_n)	0.840				BP(p_n)	0.987			
$p_n = 10$	$T_{\mathcal{E}1n}$	$T_{\mathcal{E}2n}$	$T_{\mathcal{E}3n}$	$T_{\mathcal{E}4n}$	$p_n = 11$	$T_{\mathcal{E}1n}$	$T_{\mathcal{E}2n}$	$T_{\mathcal{E}3n}$	$T_{\mathcal{E}4n}$	$p_n = 12$	$T_{\mathcal{E}1n}$	$T_{\mathcal{E}2n}$	$T_{\mathcal{E}3n}$	$T_{\mathcal{E}4n}$
TR	0.491			0.313	TR	0.796			0.407	TR	0.980			0.599
BAR	0.672	0.675	0.681	0.513	BAR	0.919	0.918	0.921	0.720	BAR	0.998	0.998	0.998	0.898
DAN	0.635	0.633	0.638	0.472	DAN	0.903	0.902	0.906	0.628	DAN	0.996	0.997	0.997	0.828
PAR	0.623	0.621	0.627	0.440	PAR	0.899	0.898	0.903	0.614	PAR	0.996	0.997	0.997	0.819
QS	0.636	0.636	0.638	0.477	QS	0.900	0.903	0.905	0.654	QS	0.996	0.997	0.997	0.852
LB(p_n)	0.498				LB(p_n)	0.803				LB(p_n)	0.981			
BP(p_n)	0.491				BP(p_n)	0.796				BP(p_n)	0.980			
$p_n = 16$	$T_{\mathcal{E}1n}$	$T_{\mathcal{E}2n}$	$T_{\mathcal{E}3n}$	$T_{\mathcal{E}4n}$	$p_n = 20$	$T_{\mathcal{E}1n}$	$T_{\mathcal{E}2n}$	$T_{\mathcal{E}3n}$	$T_{\mathcal{E}4n}$	$p_n = 24$	$T_{\mathcal{E}1n}$	$T_{\mathcal{E}2n}$	$T_{\mathcal{E}3n}$	$T_{\mathcal{E}4n}$
TR	0.429			0.239	TR	0.718			0.320	TR	0.945			0.399
BAR	0.624	0.625	0.629	0.392	BAR	0.888	0.893	0.896	0.524	BAR	0.994	0.996	0.996	0.708
DAN	0.581	0.583	0.583	0.357	DAN	0.853	0.862	0.864	0.458	DAN	0.989	0.988	0.988	0.597
PAR	0.565	0.573	0.569	0.328	PAR	0.850	0.854	0.859	0.427	PAR	0.987	0.987	0.988	0.548
QS	0.579	0.584	0.585	0.359	QS	0.853	0.863	0.867	0.470	QS	0.989	0.988	0.988	0.614
LB(p_n)	0.436				LB(p_n)	0.728				LB(p_n)	0.947			
BP(p_n)	0.429				BP(p_n)	0.718				BP(p_n)	0.945			

1) DGP: $X_t = D_t\epsilon_t, \epsilon_t \sim EXP(1), D_t = \beta_0 + \alpha X_{t-1}, \beta_0 = 1 - \alpha$. $\alpha = 0.2$ 2) 1000 iterations.

Table 2.3: Level-adjusted powers against ACD(4) at 5% level for tests of ACD effects

$n = 250$					$n = 500$					$n = 1000$				
$p_n = 6$	$T_{\mathcal{E}1n}$	$T_{\mathcal{E}2n}$	$T_{\mathcal{E}3n}$	$T_{\mathcal{E}4n}$	$p_n = 7$	$T_{\mathcal{E}1n}$	$T_{\mathcal{E}2n}$	$T_{\mathcal{E}3n}$	$T_{\mathcal{E}4n}$	$p_n = 7$	$T_{\mathcal{E}1n}$	$T_{\mathcal{E}2n}$	$T_{\mathcal{E}3n}$	$T_{\mathcal{E}4n}$
TR	0.294			0.446	TR	0.494			0.598	TR	0.759			0.816
BAR	0.349	0.335	0.334	0.536	BAR	0.607	0.591	0.591	0.768	BAR	0.839	0.832	0.828	0.944
DAN	0.346	0.328	0.335	0.492	DAN	0.608	0.596	0.593	0.722	DAN	0.840	0.826	0.825	0.915
PAR	0.345	0.329	0.330	0.525	PAR	0.609	0.591	0.591	0.725	PAR	0.836	0.823	0.822	0.924
QS	0.346	0.334	0.335	0.526	QS	0.606	0.597	0.593	0.751	QS	0.839	0.826	0.824	0.936
LB(p_n)	0.296				LB(p_n)	0.496				LB(p_n)	0.760			
BP(p_n)	0.294				BP(p_n)	0.494				BP(p_n)	0.759			
$p_n = 10$	$T_{\mathcal{E}1n}$	$T_{\mathcal{E}2n}$	$T_{\mathcal{E}3n}$	$T_{\mathcal{E}4n}$	$p_n = 11$	$T_{\mathcal{E}1n}$	$T_{\mathcal{E}2n}$	$T_{\mathcal{E}3n}$	$T_{\mathcal{E}4n}$	$p_n = 12$	$T_{\mathcal{E}1n}$	$T_{\mathcal{E}2n}$	$T_{\mathcal{E}3n}$	$T_{\mathcal{E}4n}$
TR	0.256			0.336	TR	0.427			0.454	TR	0.691			0.652
BAR	0.334	0.324	0.321	0.477	BAR	0.607	0.588	0.584	0.679	BAR	0.827	0.814	0.812	0.882
DAN	0.336	0.308	0.305	0.454	DAN	0.596	0.575	0.570	0.631	DAN	0.817	0.794	0.792	0.833
PAR	0.325	0.305	0.300	0.438	PAR	0.579	0.555	0.554	0.628	PAR	0.806	0.791	0.789	0.830
QS	0.335	0.309	0.306	0.469	QS	0.596	0.574	0.569	0.662	QS	0.816	0.792	0.791	0.860
LB(p_n)	0.259				LB(p_n)	0.428				LB(p_n)	0.692			
BP(p_n)	0.256				BP(p_n)	0.427				BP(p_n)	0.691			
$p_n = 16$	$T_{\mathcal{E}1n}$	$T_{\mathcal{E}2n}$	$T_{\mathcal{E}3n}$	$T_{\mathcal{E}4n}$	$p_n = 20$	$T_{\mathcal{E}1n}$	$T_{\mathcal{E}2n}$	$T_{\mathcal{E}3n}$	$T_{\mathcal{E}4n}$	$p_n = 24$	$T_{\mathcal{E}1n}$	$T_{\mathcal{E}2n}$	$T_{\mathcal{E}3n}$	$T_{\mathcal{E}4n}$
TR	0.229			0.259	TR	0.369			0.334	TR	0.567			0.432
BAR	0.322	0.308	0.300	0.398	BAR	0.546	0.527	0.519	0.538	BAR	0.783	0.766	0.760	0.723
DAN	0.308	0.290	0.283	0.371	DAN	0.504	0.489	0.474	0.489	DAN	0.737	0.716	0.706	0.639
PAR	0.304	0.283	0.269	0.342	PAR	0.506	0.481	0.470	0.462	PAR	0.727	0.704	0.695	0.607
QS	0.306	0.294	0.285	0.374	QS	0.508	0.491	0.470	0.505	QS	0.734	0.717	0.714	0.650
LB(p_n)	0.231				LB(p_n)	0.371				LB(p_n)	0.574			
BP(p_n)	0.229				BP(p_n)	0.369				BP(p_n)	0.567			

1) DGP: $X_t = D_t\epsilon_t$, $D_t = \beta_0 + \sum_{i=1}^{4}\alpha_i X_{t-i}$, $\beta_0 = 1 - \sum_{i=1}^{4}\alpha_i$, $\alpha_i = \alpha(1-\frac{i}{5})$, $\alpha = \frac{0.2}{\sum_{i=1}^{4}(1-\frac{i}{5})}$, $\epsilon_t \sim EXP(1)$. 2) 1000 iterations.

Table 2.4: Level-adjusted powers against ACD(12) at 5% level for tests of ACD effects

	$n=250$					$n=500$					$n=1000$			
$p_n=6$	$T_{\mathcal{E}1n}$	$T_{\mathcal{E}2n}$	$T_{\mathcal{E}3n}$	$T_{\mathcal{E}4n}$	$p_n=7$	$T_{\mathcal{E}1n}$	$T_{\mathcal{E}2n}$	$T_{\mathcal{E}3n}$	$T_{\mathcal{E}4n}$	$p_n=7$	$T_{\mathcal{E}1n}$	$T_{\mathcal{E}2n}$	$T_{\mathcal{E}3n}$	$T_{\mathcal{E}4n}$
TR	0.558			0.764	TR	0.867			0.961	TR	0.988			0.997
BAR	0.445	0.422	0.414	0.666	BAR	0.789	0.766	0.755	0.929	BAR	0.954	0.947	0.945	0.993
DAN	0.472	0.445	0.450	0.541	DAN	0.808	0.800	0.794	0.837	DAN	0.972	0.967	0.965	0.973
PAR	0.487	0.455	0.448	0.738	PAR	0.826	0.801	0.787	0.952	PAR	0.973	0.967	0.966	0.997
QS	0.463	0.437	0.431	0.658	QS	0.802	0.784	0.776	0.919	QS	0.967	0.959	0.955	0.992
LB(p_n)	0.559				LB(p_n)	0.867				LB(p_n)	0.988			
BP(p_n)	0.558				BP(p_n)	0.867				BP(p_n)	0.988			
$p_n=10$	$T_{\mathcal{E}1n}$	$T_{\mathcal{E}2n}$	$T_{\mathcal{E}3n}$	$T_{\mathcal{E}4n}$	$p_n=11$	$T_{\mathcal{E}1n}$	$T_{\mathcal{E}2n}$	$T_{\mathcal{E}3n}$	$T_{\mathcal{E}4n}$	$p_n=12$	$T_{\mathcal{E}1n}$	$T_{\mathcal{E}2n}$	$T_{\mathcal{E}3n}$	$T_{\mathcal{E}4n}$
TR	0.644			0.849	TR	0.914			0.981	TR	0.999			1.000
BAR	0.539	0.496	0.485	0.788	BAR	0.862	0.830	0.823	0.966	BAR	0.985	0.979	0.978	0.999
DAN	0.551	0.512	0.500	0.732	DAN	0.869	0.844	0.836	0.945	DAN	0.988	0.985	0.981	0.999
PAR	0.574	0.525	0.504	0.826	PAR	0.884	0.857	0.852	0.979	PAR	0.994	0.990	0.990	0.999
QS	0.552	0.509	0.497	0.795	QS	0.875	0.849	0.834	0.973	QS	0.990	0.985	0.981	0.999
LB(p_n)	0.644				LB(p_n)	0.914				LB(p_n)	0.999			
BP(p_n)	0.644				BP(p_n)	0.914				BP(p_n)	0.999			
$p_n=16$	$T_{\mathcal{E}1n}$	$T_{\mathcal{E}2n}$	$T_{\mathcal{E}3n}$	$T_{\mathcal{E}4n}$	$p_n=20$	$T_{\mathcal{E}1n}$	$T_{\mathcal{E}2n}$	$T_{\mathcal{E}3n}$	$T_{\mathcal{E}4n}$	$p_n=24$	$T_{\mathcal{E}1n}$	$T_{\mathcal{E}2n}$	$T_{\mathcal{E}3n}$	$T_{\mathcal{E}4n}$
TR	0.642			0.848	TR	0.908			0.976	TR	0.996			0.999
BAR	0.600	0.554	0.532	0.850	BAR	0.902	0.888	0.880	0.987	BAR	0.998	0.995	0.995	0.999
DAN	0.616	0.567	0.541	0.841	DAN	0.910	0.887	0.877	0.984	DAN	0.998	0.995	0.994	0.999
PAR	0.632	0.575	0.544	0.865	PAR	0.917	0.897	0.882	0.985	PAR	0.998	0.995	0.995	0.999
QS	0.620	0.565	0.541	0.858	QS	0.912	0.889	0.879	0.989	QS	0.998	0.995	0.995	0.999
LB(p_n)	0.640				LB(p_n)	0.908				LB(p_n)	0.996			
BP(p_n)	0.642				BP(p_n)	0.908				BP(p_n)	0.996			

1) DGP: $X_t = D_t\epsilon_t$, $D_t = \beta_0 + \sum_{i=1}^{12} \alpha_i X_{t-i}, \beta_0 = 1 - \sum_{i=1}^{12} \alpha_i, \alpha_i = 0.05, \epsilon_t \sim EXP(1)$. 2) 1000 iterations.

Table 2.5: Level-adjusted powers against ACD(1,1) at 5% level for tests of ACD effects

$n = 250$					$n = 500$					$n = 1000$				
$p_n = 6$	$T_{\mathcal{E}1n}$	$T_{\mathcal{E}2n}$	$T_{\mathcal{E}3n}$	$T_{\mathcal{E}4n}$	$p_n = 7$	$T_{\mathcal{E}1n}$	$T_{\mathcal{E}2n}$	$T_{\mathcal{E}3n}$	$T_{\mathcal{E}4n}$	$p_n = 7$	$T_{\mathcal{E}1n}$	$T_{\mathcal{E}2n}$	$T_{\mathcal{E}3n}$	$T_{\mathcal{E}4n}$
TR	0.424			0.438	TR	0.705			0.583	TR	0.930			0.796
BAR	0.566	0.562	0.561	0.610	BAR	0.850	0.845	0.847	0.829	BAR	0.981	0.981	0.984	0.964
DAN	0.546	0.532	0.548	0.564	DAN	0.842	0.841	0.840	0.775	DAN	0.980	0.976	0.978	0.947
PAR	0.530	0.524	0.531	0.573	PAR	0.832	0.827	0.828	0.769	PAR	0.973	0.972	0.974	0.950
QS	0.542	0.535	0.551	0.587	QS	0.841	0.840	0.840	0.805	QS	0.977	0.975	0.977	0.955
LB(p_n)	0.430				LB(p_n)	0.706				LB(p_n)	0.930			
BP(p_n)	0.424				BP(p_n)	0.705				BP(p_n)	0.930			
$p_n = 10$	$T_{\mathcal{E}1n}$	$T_{\mathcal{E}2n}$	$T_{\mathcal{E}3n}$	$T_{\mathcal{E}4n}$	$p_n = 11$	$T_{\mathcal{E}1n}$	$T_{\mathcal{E}2n}$	$T_{\mathcal{E}3n}$	$T_{\mathcal{E}4n}$	$p_n = 12$	$T_{\mathcal{E}1n}$	$T_{\mathcal{E}2n}$	$T_{\mathcal{E}3n}$	$T_{\mathcal{E}4n}$
TR	0.372			0.324	TR	0.640			0.426	TR	0.896			0.629
BAR	0.522	0.512	0.515	0.511	BAR	0.826	0.819	0.824	0.708	BAR	0.969	0.970	0.971	0.899
DAN	0.506	0.484	0.485	0.479	DAN	0.798	0.789	0.793	0.639	DAN	0.963	0.961	0.963	0.837
PAR	0.501	0.475	0.477	0.455	PAR	0.786	0.778	0.780	0.627	PAR	0.962	0.960	0.962	0.833
QS	0.504	0.488	0.486	0.491	QS	0.798	0.787	0.788	0.671	QS	0.964	0.961	0.963	0.862
LB(p_n)	0.377				LB(p_n)	0.645				LB(p_n)	0.896			
BP(p_n)	0.372				BP(p_n)	0.640				BP(p_n)	0.896			
$p_n = 16$	$T_{\mathcal{E}1n}$	$T_{\mathcal{E}2n}$	$T_{\mathcal{E}3n}$	$T_{\mathcal{E}4n}$	$p_n = 20$	$T_{\mathcal{E}1n}$	$T_{\mathcal{E}2n}$	$T_{\mathcal{E}3n}$	$T_{\mathcal{E}4n}$	$p_n = 24$	$T_{\mathcal{E}1n}$	$T_{\mathcal{E}2n}$	$T_{\mathcal{E}3n}$	$T_{\mathcal{E}4n}$
TR	0.313			0.251	TR	0.559			0.326	TR	0.819			0.417
BAR	0.497	0.481	0.479	0.405	BAR	0.773	0.772	0.775	0.534	BAR	0.952	0.948	0.950	0.721
DAN	0.455	0.435	0.425	0.366	DAN	0.728	0.727	0.722	0.482	DAN	0.933	0.924	0.919	0.633
PAR	0.439	0.423	0.416	0.345	PAR	0.724	0.718	0.716	0.448	PAR	0.930	0.918	0.913	0.581
QS	0.453	0.434	0.425	0.369	QS	0.729	0.728	0.722	0.493	QS	0.933	0.923	0.925	0.638
LB(p_n)	0.322				LB(p_n)	0.563				LB(p_n)	0.821			
BP(p_n)	0.313				BP(p_n)	0.559				BP(p_n)	0.819			

1) DGP: $X_t = D_t \epsilon_t$, $D_t = \beta_0 + \alpha X_{t-1} + \beta D_{t-1}$,$\beta_0 = 1 - \alpha - \beta, \epsilon_t \sim EXP(1)$ $\alpha = 0.15, \beta = 0.25$. 2) 1000 iterations.

Table 2.6: Empirical levels at the 5% level for tests of adjustment when the model is ACD(1,1)

$n=250$				$n=500$				$n=1000$			
$p_n=6$	$T_{\mathcal{A}1n}$	$T_{\mathcal{A}2n}$	$T_{\mathcal{A}3n}$	$p_n=7$	$T_{\mathcal{A}1n}$	$T_{\mathcal{A}2n}$	$T_{\mathcal{A}3n}$	$p_n=7$	$T_{\mathcal{A}1n}$	$T_{\mathcal{A}2n}$	$T_{\mathcal{A}3n}$
TR	0.0470			TR	0.0410			TR	0.0406		
BAR	0.0296	0.0288	0.0278	BAR	0.0272	0.0268	0.0236	BAR	0.0312	0.0318	0.0310
DAN	0.0312	0.0306	0.0310	DAN	0.0292	0.0284	0.0268	DAN	0.0326	0.0330	0.0336
PAR	0.0332	0.0342	0.0332	PAR	0.0294	0.0298	0.0266	PAR	0.0334	0.0338	0.0340
QS	0.0322	0.0318	0.0314	QS	0.0286	0.0292	0.0272	QS	0.0328	0.0340	0.0328
LB(p_n)	0.0306			LB(p_n)	0.0282			LB(p_n)	0.0260		
BP(p_n)	0.0334			BP(p_n)	0.0292			BP(p_n)	0.0268		
$p_n=10$	$T_{\mathcal{A}1n}$	$T_{\mathcal{A}2n}$	$T_{\mathcal{A}3n}$	$p_n=11$	$T_{\mathcal{A}1n}$	$T_{\mathcal{A}2n}$	$T_{\mathcal{A}3n}$	$p_n=12$	$T_{\mathcal{A}1n}$	$T_{\mathcal{A}2n}$	$T_{\mathcal{A}3n}$
TR	0.0548			TR	0.0468			TR	0.0464		
BAR	0.0376	0.0364	0.0372	BAR	0.0336	0.0286	0.0280	BAR	0.0340	0.0336	0.0354
DAN	0.0418	0.0436	0.0468	DAN	0.0356	0.0328	0.0344	DAN	0.0370	0.0360	0.0376
PAR	0.0436	0.0426	0.0440	PAR	0.0378	0.0334	0.0346	PAR	0.0364	0.0356	0.0400
QS	0.0418	0.0428	0.0458	QS	0.0368	0.0324	0.0334	QS	0.0366	0.0358	0.0386
LB(p_n)	0.0272			LB(p_n)	0.0304			LB(p_n)	0.0294		
BP(p_n)	0.0326			BP(p_n)	0.0338			BP(p_n)	0.0310		
$p_n=16$	$T_{\mathcal{A}1n}$	$T_{\mathcal{A}2n}$	$T_{\mathcal{A}3n}$	$p_n=20$	$T_{\mathcal{A}1n}$	$T_{\mathcal{A}2n}$	$T_{\mathcal{A}3n}$	$p_n=24$	$T_{\mathcal{A}1n}$	$T_{\mathcal{A}2n}$	$T_{\mathcal{A}3n}$
TR	0.0588			TR	0.0496			TR	0.0438		
BAR	0.0434	0.0384	0.0402	BAR	0.0380	0.0338	0.0334	BAR	0.0390	0.0368	0.0384
DAN	0.0466	0.0442	0.0482	DAN	0.0408	0.0384	0.0406	DAN	0.0420	0.0398	0.0428
PAR	0.0474	0.0430	0.0460	PAR	0.0416	0.0368	0.0396	PAR	0.0416	0.0392	0.0422
QS	0.0476	0.0428	0.0470	QS	0.0408	0.0368	0.0414	QS	0.0426	0.0394	0.0420
LB(p_n)	0.0296			LB(p_n)	0.0284			LB(p_n)	0.0276		
BP(p_n)	0.0378			BP(p_n)	0.0322			BP(p_n)	0.0306		

1) DGP: $X_t = D_t\epsilon_t,\ D_t = \beta_0 + \alpha X_{t-1} + \beta D_{t-1}, \beta_0 = 1-\alpha-\beta, \alpha = 0.18, \beta = 0.80$,where ϵ_t is $EXP(1)$. 2) 5000 iterations.

Table 2.7: Level-adjusted powers against ACD(2,1) at 5% level for tests of adjustment when the model is ACD(1,1)

$n = 250$				$n = 500$				$n = 1000$			
$p_n = 6$	$T_{\mathcal{A}1n}$	$T_{\mathcal{A}2n}$	$T_{\mathcal{A}3n}$	$p_n = 7$	$T_{\mathcal{A}1n}$	$T_{\mathcal{A}2n}$	$T_{\mathcal{A}3n}$	$p_n = 7$	$T_{\mathcal{A}1n}$	$T_{\mathcal{A}2n}$	$T_{\mathcal{A}3n}$
TR	0.433			TR	0.764			TR	0.976		
BAR	0.610	0.599	0.580	BAR	0.884	0.885	0.870	BAR	0.994	0.994	0.992
DAN	0.611	0.607	0.588	DAN	0.886	0.888	0.873	DAN	0.995	0.995	0.993
PAR	0.602	0.593	0.564	PAR	0.880	0.880	0.872	PAR	0.995	0.994	0.992
QS	0.615	0.611	0.585	QS	0.886	0.888	0.875	QS	0.995	0.995	0.992
LB(p_n)	0.437			LB(p_n)	0.765			LB(p_n)	0.976		
BP(p_n)	0.433			BP(p_n)	0.764			BP(p_n)	0.976		
$p_n = 10$	$T_{\mathcal{A}1n}$	$T_{\mathcal{A}2n}$	$T_{\mathcal{A}3n}$	$p_n = 11$	$T_{\mathcal{A}1n}$	$T_{\mathcal{A}2n}$	$T_{\mathcal{A}3n}$	$p_n = 12$	$T_{\mathcal{A}1n}$	$T_{\mathcal{A}2n}$	$T_{\mathcal{A}3n}$
TR	0.322			TR	0.657			TR	0.935		
BAR	0.565	0.542	0.511	BAR	0.845	0.847	0.831	BAR	0.991	0.990	0.988
DAN	0.529	0.511	0.484	DAN	0.828	0.822	0.811	DAN	0.989	0.988	0.983
PAR	0.506	0.495	0.465	PAR	0.815	0.814	0.802	PAR	0.985	0.985	0.980
QS	0.529	0.512	0.480	QS	0.822	0.821	0.812	QS	0.987	0.988	0.983
LB(p_n)	0.326			LB(p_n)	0.660			LB(p_n)	0.937		
BP(p_n)	0.322			BP(p_n)	0.657			BP(p_n)	0.935		
$p_n = 16$	$T_{\mathcal{A}1n}$	$T_{\mathcal{A}2n}$	$T_{\mathcal{A}3n}$	$p_n = 20$	$T_{\mathcal{A}1n}$	$T_{\mathcal{A}2n}$	$T_{\mathcal{A}3n}$	$p_n = 24$	$T_{\mathcal{A}1n}$	$T_{\mathcal{A}2n}$	$T_{\mathcal{A}3n}$
TR	0.281			TR	0.571			TR	0.882		
BAR	0.509	0.488	0.464	BAR	0.816	0.812	0.798	BAR	0.980	0.979	0.973
DAN	0.444	0.438	0.415	DAN	0.785	0.780	0.756	DAN	0.966	0.966	0.959
PAR	0.430	0.438	0.405	PAR	0.770	0.771	0.749	PAR	0.961	0.958	0.955
QS	0.447	0.446	0.417	QS	0.784	0.776	0.760	QS	0.963	0.966	0.957
LB(p_n)	0.285			LB(p_n)	0.579			LB(p_n)	0.883		
BP(p_n)	0.281			BP(p_n)	0.571			BP(p_n)	0.882		

1) DGP: $\mathrm{X}_t = D_t \epsilon_t, \epsilon_t \sim EXP(1), D_t = \beta_0 + \alpha_1 X_{t-1} + \alpha_2 X_{t-2} + \beta_1 D_{t-1}$, $\beta_0 = 1 - \alpha_1 - \alpha_2 - \beta_1, \alpha_1 = 0.3, \alpha_2 = 0.4, \beta_1 = 0.1$. 2) 1000 iterations.

Table 2.8: Level-adjusted powers against ACD(2,2)[a] at 5% level for tests of adjustment when the model is ACD(1,1)

$n = 250$				$n = 500$				$n = 1000$			
$p_n = 6$	T_{A1n}	T_{A2n}	T_{A3n}	$p_n = 7$	T_{A1n}	T_{A2n}	T_{A3n}	$p_n = 7$	T_{A1n}	T_{A2n}	T_{A3n}
TR	0.487			TR	0.754			TR	0.962		
BAR	0.637	0.617	0.583	BAR	0.866	0.859	0.836	BAR	0.993	0.993	0.988
DAN	0.629	0.611	0.580	DAN	0.868	0.857	0.833	DAN	0.994	0.993	0.985
PAR	0.625	0.601	0.563	PAR	0.863	0.851	0.825	PAR	0.992	0.991	0.984
QS	0.633	0.609	0.584	QS	0.869	0.855	0.830	QS	0.994	0.993	0.987
LB(p_n)	0.489			LB(p_n)	0.754			LB(p_n)	0.962		
BP(p_n)	0.487			BP(p_n)	0.754			BP(p_n)	0.962		
$p_n = 10$	T_{A1n}	T_{A2n}	T_{A3n}	$p_n = 11$	T_{A1n}	T_{A2n}	T_{A3n}	$p_n = 12$	T_{A1n}	T_{A2n}	T_{A3n}
TR	0.377			TR	0.657			TR	0.919		
BAR	0.606	0.560	0.521	BAR	0.843	0.826	0.793	BAR	0.987	0.982	0.974
DAN	0.564	0.533	0.490	DAN	0.819	0.801	0.775	DAN	0.977	0.975	0.967
PAR	0.547	0.527	0.470	PAR	0.803	0.792	0.755	PAR	0.975	0.971	0.964
QS	0.573	0.534	0.485	QS	0.815	0.793	0.775	QS	0.978	0.974	0.966
LB(p_n)	0.379			LB(p_n)	0.662			LB(p_n)	0.920		
BP(p_n)	0.377			BP(p_n)	0.657			BP(p_n)	0.919		
$p_n = 16$	T_{A1n}	T_{A2n}	T_{A3n}	$p_n = 20$	T_{A1n}	T_{A2n}	T_{A3n}	$p_n = 24$	T_{A1n}	T_{A2n}	T_{A3n}
TR	0.327			TR	0.576			TR	0.869		
BAR	0.544	0.519	0.476	BAR	0.798	0.786	0.755	BAR	0.969	0.967	0.954
DAN	0.491	0.466	0.422	DAN	0.767	0.744	0.711	DAN	0.956	0.953	0.936
PAR	0.480	0.455	0.416	PAR	0.760	0.740	0.706	PAR	0.953	0.948	0.934
QS	0.490	0.470	0.427	QS	0.767	0.746	0.714	QS	0.955	0.949	0.940
LB(p_n)	0.331			LB(p_n)	0.583			LB(p_n)	0.872		
BP(p_n)	0.327			BP(p_n)	0.576			BP(p_n)	0.869		

1) DGP: $\mathrm{X}_t = D_t\epsilon_t, \epsilon_t \sim EXP(1), D_t = \beta_0 + \alpha_1 X_{t-1} + \alpha_2 X_{t-2} + \beta_1 D_{t-1} + \beta_2 D_{t-2}$, $\beta_0 = 1 - \alpha_1 - \alpha_2 - \beta_1 - \beta_2, \alpha_1 = 0.1, \alpha_2 = 0.3, \beta_1 = 0.1, \beta_2 = 0.3$. 2) 1000 iterations.

Table 2.9: Level-adjusted powers against ACD(2,2)[b] at 5% level for tests of adjustment when the model is ACD(1,1)

$n=250$				$n=500$				$n=1000$			
$p_n=6$	T_{A1n}	T_{A2n}	T_{A3n}	$p_n=7$	T_{A1n}	T_{A2n}	T_{A3n}	$p_n=7$	T_{A1n}	T_{A2n}	T_{A3n}
TR	0.222			TR	0.423			TR	0.699		
BAR	0.332	0.315	0.292	BAR	0.563	0.559	0.530	BAR	0.833	0.824	0.783
DAN	0.340	0.327	0.311	DAN	0.570	0.563	0.535	DAN	0.833	0.829	0.795
PAR	0.329	0.318	0.297	PAR	0.557	0.552	0.529	PAR	0.832	0.821	0.784
QS	0.339	0.328	0.313	QS	0.574	0.564	0.536	QS	0.834	0.827	0.797
LB(p_n)	0.224			LB(p_n)	0.422			LB(p_n)	0.699		
BP(p_n)	0.222			BP(p_n)	0.423			BP(p_n)	0.699		
$p_n=10$	T_{A1n}	T_{A2n}	T_{A3n}	$p_n=11$	T_{A1n}	T_{A2n}	T_{A3n}	$p_n=12$	T_{A1n}	T_{A2n}	T_{A3n}
TR	0.168			TR	0.331			TR	0.566		
BAR	0.292	0.276	0.260	BAR	0.525	0.516	0.486	BAR	0.796	0.781	0.757
DAN	0.275	0.265	0.238	DAN	0.496	0.492	0.461	DAN	0.774	0.767	0.719
PAR	0.259	0.255	0.227	PAR	0.483	0.474	0.445	PAR	0.754	0.746	0.697
QS	0.275	0.267	0.235	QS	0.494	0.485	0.459	QS	0.772	0.767	0.716
LB(p_n)	0.166			LB(p_n)	0.336			LB(p_n)	0.568		
BP(p_n)	0.168			BP(p_n)	0.331			BP(p_n)	0.566		
$p_n=16$	T_{A1n}	T_{A2n}	T_{A3n}	$p_n=20$	T_{A1n}	T_{A2n}	T_{A3n}	$p_n=24$	T_{A1n}	T_{A2n}	T_{A3n}
TR	0.138			TR	0.280			TR	0.473		
BAR	0.259	0.245	0.227	BAR	0.470	0.471	0.437	BAR	0.722	0.715	0.669
DAN	0.234	0.229	0.209	DAN	0.441	0.430	0.401	DAN	0.667	0.660	0.614
PAR	0.228	0.225	0.209	PAR	0.439	0.422	0.393	PAR	0.647	0.630	0.599
QS	0.236	0.230	0.210	QS	0.444	0.424	0.396	QS	0.657	0.661	0.611
LB(p_n)	0.138			LB(p_n)	0.285			LB(p_n)	0.476		
BP(p_n)	0.138			BP(p_n)	0.280			BP(p_n)	0.473		

1) DGP: $X_t = D_t\epsilon_t, \epsilon_t \sim EXP(1), D_t = \beta_0 + \alpha_1 X_{t-1} + \alpha_2 X_{t-2} + \beta_1 D_{t-1} + \beta_2 D_{t-2}$, $\beta_0 = 1 - \alpha_1 - \alpha_2 - \beta_1 - \beta_2, \alpha_1 = 0.1, \alpha_2 = 0.2, \beta_1 = 0, \beta_2 = 0.1$. 2) 1000 iterations.

Table 2.10: Level-adjusted powers against ACD(4,4) at 5% level for tests of adjustment when the model is ACD(1,1)

$n = 250$				$n = 500$				$n = 1000$			
$p_n = 6$	T_{A1n}	T_{A2n}	T_{A3n}	$p_n = 7$	T_{A1n}	T_{A2n}	T_{A3n}	$p_n = 7$	T_{A1n}	T_{A2n}	T_{A3n}
TR	0.544			TR	0.825			TR	0.989		
BAR	0.262	0.254	0.236	BAR	0.681	0.682	0.648	BAR	0.962	0.960	0.946
DAN	0.276	0.280	0.270	DAN	0.728	0.727	0.711	DAN	0.969	0.969	0.960
PAR	0.371	0.368	0.334	PAR	0.775	0.774	0.757	PAR	0.984	0.984	0.977
QS	0.292	0.288	0.272	QS	0.738	0.740	0.717	QS	0.976	0.975	0.964
LB(p_n)	0.543			LB(p_n)	0.824			LB(p_n)	0.989		
BP(p_n)	0.544			BP(p_n)	0.825			BP(p_n)	0.989		
$p_n = 10$	T_{A1n}	T_{A2n}	T_{A3n}	$p_n = 11$	T_{A1n}	T_{A2n}	T_{A3n}	$p_n = 12$	T_{A1n}	T_{A2n}	T_{A3n}
TR	0.436			TR	0.757			TR	0.975		
BAR	0.483	0.468	0.449	BAR	0.809	0.813	0.793	BAR	0.988	0.988	0.986
DAN	0.513	0.508	0.486	DAN	0.828	0.832	0.816	DAN	0.990	0.990	0.988
PAR	0.522	0.514	0.486	PAR	0.830	0.834	0.813	PAR	0.990	0.989	0.987
QS	0.520	0.516	0.485	QS	0.830	0.833	0.815	QS	0.990	0.990	0.987
LB(p_n)	0.436			LB(p_n)	0.759			LB(p_n)	0.975		
BP(p_n)	0.436			BP(p_n)	0.757			BP(p_n)	0.975		
$p_n = 16$	T_{A1n}	T_{A2n}	T_{A3n}	$p_n = 20$	T_{A1n}	T_{A2n}	T_{A3n}	$p_n = 24$	T_{A1n}	T_{A2n}	T_{A3n}
TR	0.396			TR	0.661			TR	0.935		
BAR	0.503	0.495	0.478	BAR	0.819	0.821	0.805	BAR	0.987	0.987	0.984
DAN	0.518	0.521	0.497	DAN	0.819	0.823	0.806	DAN	0.987	0.985	0.983
PAR	0.516	0.508	0.488	PAR	0.816	0.818	0.800	PAR	0.987	0.984	0.982
QS	0.516	0.517	0.492	QS	0.821	0.821	0.805	QS	0.987	0.985	0.983
LB(p_n)	0.399			LB(p_n)	0.671			LB(p_n)	0.936		
BP(p_n)	0.396			BP(p_n)	0.661			BP(p_n)	0.935		

1) DGP: $X_t = D_t\epsilon_t, \epsilon_t \sim EXP(1), D_t = \beta_0 + \sum_{i=1}^{4}\alpha_i X_{t-i} + \sum_{i=1}^{4}\beta_i D_{t-i}$, $\beta_0 = 1 - \sum_{i=1}^{4}(\alpha_i+\beta_i), \alpha_1 = \alpha_2 = \alpha_3 = \beta_1 = \beta_2 = \beta_3 = 0, \alpha_4 = 0.2, \beta_4 = 0.1$ 2) 1000 iterations.

Table 2.11: Descriptive statistics of IBM durations (in seconds)

	Obs	Mean	Minimum	Maximum	Coef. of variation
Raw trade durations	52145	26.64	1	561	1.374
Seasonally adj. trade durations	52145	0.9994	0.0271	23.86	1.325
Raw volume durations	1025	1313	2	9312	0.743
Seasonally adj. volume durations	1025	0.9925	0.0001	5.06	0.621

Table 2.12: Test statistics for ACD effects for IBM trade durations

p_n	$T_{\mathcal{E}1n}(k_{BAR})$		$T_{\mathcal{E}1n}(k_{DAN})$		LB	
	Raw durations	Adj. durations	Raw durations	Adj. durations	Raw durations	Adj. durations
6	1463.973	1012.807	1590.516	1095.242	5966.923	4010.247
	(0^+)	(0^+)	(0^+)	(0^+)	(0^+)	(0^+)
10	1750.530	1193.901	1878.026	1272.329	9193.600	6095.899
	(0^+)	(0^+)	(0^+)	(0^+)	(0^+)	(0^+)
16	2047.776	1375.893	2186.136	1458.629	12929.797	8409.842
	(0^+)	(0^+)	(0^+)	(0^+)	(0^+)	(0^+)
24	2309.805	1532.195	2438.914	1607.834	17639.104	11375.798
	(0^+)	(0^+)	(0^+)	(0^+)	(0^+)	(0^+)

Note: The notation 0^+ means a number smaller than 10^{-4}.

Table 2.13: Test statistics for ACD effects for IBM volume durations

p_n	$T_{\mathcal{E}1n}(k_{BAR})$		$T_{\mathcal{E}1n}(k_{DAN})$		LB	
	Raw durations	Adj. durations	Raw durations	Adj. durations	Raw durations	Adj. durations
7	208.518	231.741	204.882	237.708	553.079	860.026
	(0^+)	(0^+)	(0^+)	(0^+)	(0^+)	(0^+)
14	190.186	259.167	178.597	259.342	792.689	1297.077
	(0^+)	(0^+)	(0^+)	(0^+)	(0^+)	(0^+)
24	180.657	272.808	167.926	268.966	989.484	1597.511
	(0^+)	(0^+)	(0^+)	(0^+)	(0^+)	(0^+)

Note: The notation 0^+ means a number smaller than 10^{-4}.

Table 2.14: Test statistics for adjustment of EACD(1,1) and EACD(2,2) models for IBM trade durations

	EACD(1,1)			EACD(2,2)		
p_n	$T_{\mathcal{A}1n}(k_{TR})$	$T_{\mathcal{A}1n}(k_{BAR})$	$T_{\mathcal{A}1n}(k_{DAN})$	$T_{\mathcal{A}1n}(k_{TR})$	$T_{\mathcal{A}1n}(k_{BAR})$	$T_{\mathcal{A}1n}(k_{DAN})$
6	4.094	6.610	6.447	2.587	0.341	0.380
	(0^+)	(0^+)	(0^+)	(0.005)	(0.366)	(0.352)
10	2.756	5.577	5.484	1.516	1.197	1.566
	(0.003)	(0^+)	(0^+)	(0.065)	(0.116)	(0.059)
16	6.303	4.780	4.305	2.748	1.606	1.756
	(0^+)	(0^+)	(0^+)	(0.003)	(0.054)	(0.039)
24	6.041	5.213	5.218	1.432	2.012	2.219
	(0^+)	(0^+)	(0^+)	(0.076)	(0.022)	(0.013)

Note: The notation 0^+ means a number smaller than 10^{-4}.

Table 2.15: Test statistics for adjustment of WACD(1,1) and WACD(2,2) models for IBM trade durations

	WACD(1,1)			WACD(2,2)		
p_n	$T_{\mathcal{A}1n}(k_{TR})$	$T_{\mathcal{A}1n}(k_{BAR})$	$T_{\mathcal{A}1n}(k_{DAN})$	$T_{\mathcal{A}1n}(k_{TR})$	$T_{\mathcal{A}1n}(k_{BAR})$	$T_{\mathcal{A}1n}(k_{DAN})$
6	3.970	6.282	6.127	2.716	0.343	0.394
	(0^+)	(0^+)	(0^+)	(0.003)	(0.366)	(0.347)
10	2.679	5.320	5.250	1.648	1.257	1.651
	(0.004)	(0^+)	(0^+)	(0.050)	(0.104)	(0.049)
16	6.243	4.593	4.148	2.970	1.714	1.881
	(0^+)	(0^+)	(0^+)	(0.001)	(0.043)	(0.030)
24	5.929	5.070	5.097	1.615	2.166	2.393
	(0^+)	(0^+)	(0^+)	(0.053)	(0.015)	(0.008)

Note: The notation 0^+ means a number smaller than 10^{-4}.

Table 2.16: Test statistics for adjustment of GACD(1,1) and GACD(2,2) models for IBM trade durations

	GACD(1,1)			GACD(2,2)		
p_n	$T_{\mathcal{A}1n}(k_{TR})$	$T_{\mathcal{A}1n}(k_{BAR})$	$T_{\mathcal{A}1n}(k_{DAN})$	$T_{\mathcal{A}1n}(k_{TR})$	$T_{\mathcal{A}1n}(k_{BAR})$	$T_{\mathcal{A}1n}(k_{DAN})$
6	3.871	6.507	6.275	2.157	0.719	0.692
	(0^+)	(0^+)	(0^+)	(0.015)	(0.236)	(0.244)
10	2.481	5.424	5.170	1.147	1.201	1.432
	(0.006)	(0^+)	(0^+)	(0.126)	(0.115)	(0.076)
16	4.978	4.508	3.900	2.384	1.396	1.431
	(0^+)	(0^+)	(0^+)	(0.008)	(0.081)	(0.076)
24	4.002	4.549	4.310	1.133	1.708	1.834
	(0^+)	(0^+)	(0^+)	(0.128)	(0.044)	(0.033)

Note: The notation 0^+ means a number smaller than 10^{-4}.

Table 2.17: Test statistics for adjustment of EACD(1,1) and EACD(2,2) models for IBM volume durations

	EACD(1,1)			EACD(2,2)		
p_n	$T_{\mathcal{A}1n}(k_{TR})$	$T_{\mathcal{A}1n}(k_{BAR})$	$T_{\mathcal{A}1n}(k_{DAN})$	$T_{\mathcal{A}1n}(k_{TR})$	$T_{\mathcal{A}1n}(k_{BAR})$	$T_{\mathcal{A}1n}(k_{DAN})$
7	1.520	2.643	2.501	-1.420	-0.963	-1.060
	(0.064)	(0.004)	(0.006)	(0.922)	(0.832)	(0.855)
14	0.573	2.130	1.889	-1.590	-1.390	-1.470
	(0.283)	(0.016)	(0.029)	(0.944)	(0.918)	(0.930)
24	-0.022	1.437	1.121	-1.794	-1.693	-1.745
	(0.509)	(0.075)	(0.131)	(0.963)	(0.955)	(0.960)

Note: The notation 0^+ means a number smaller than 10^{-4}.

Table 2.18: Test statistics for adjustment of WACD(1,1) and WACD(2,2) models for IBM volume durations

	WACD(1,1)			WACD(2,2)		
p_n	$T_{\mathcal{A}1n}(k_{TR})$	$T_{\mathcal{A}1n}(k_{BAR})$	$T_{\mathcal{A}1n}(k_{DAN})$	$T_{\mathcal{A}1n}(k_{TR})$	$T_{\mathcal{A}1n}(k_{BAR})$	$T_{\mathcal{A}1n}(k_{DAN})$
7	2.539	5.708	5.139	-0.423	1.056	0.789
	(0.005)	(0^+)	(0^+)	(0.664)	(0.145)	(0.215)
14	1.313	4.084	3.387	-0.739	0.175	-0.131
	(0.094)	(0^+)	(0.0003)	(0.770)	(0.431)	(0.552)
24	0.465	2.825	2.167	-1.206	-0.407	-0.643
	(0.321)	(0.002)	(0.015)	(0.886)	(0.658)	(0.740)

Note: The notation 0^+ means a number smaller than 10^{-4}.

Table 2.19: Test statistics for adjustment of GACD(1,1) and GACD(2,2) models for IBM volume durations

	GACD(1,1)			GACD(2,2)		
p_n	$T_{\mathcal{A}1n}(k_{TR})$	$T_{\mathcal{A}1n}(k_{BAR})$	$T_{\mathcal{A}1n}(k_{DAN})$	$T_{\mathcal{A}1n}(k_{TR})$	$T_{\mathcal{A}1n}(k_{BAR})$	$T_{\mathcal{A}1n}(k_{DAN})$
7	3.354	7.725	6.948	0.419	2.585	2.217
	(0.0004)	(0^+)	(0^+)	(0.337)	(0.005)	(0.013)
14	1.917	5.476	4.518	-0.044	1.409	0.953
	(0.028)	(0^+)	(0^+)	(0.518)	(0.079)	(0.170)
24	0.886	3.865	2.991	-0.692	0.624	0.252
	(0.188)	(0^+)	(0.001)	(0.756)	(0.266)	(0.400)

Note: The notation 0^+ means a number smaller than 10^{-4}.

Chapter 3

Intraday Value at Risk (IVaR) using tick-by-tick data with application to the Toronto Stock Exchange

3.1 Introduction

Our objective is to propose an Intraday Value at Risk (IVaR) based on tick-by-tick data by extending the VaR techniques to intraday data. VaR refers to the maximum expected loss that will not be exceeded under normal market conditions over a predetermined period at a given confidence level (Jorion, 2001, p.xxii). In other words, VaR corresponds to the quantile of the conditional distribution of price changes over a target horizon and for a certain confidence level. Financial institutions generally produce their market VaR at the end of the business day to measure their total risk exposure over the next day. For regulated capital adequacy purposes, banks usually compute the market VaR daily and then re-scale it to a 10-day horizon. Today, VaR has been largely adopted by financial institutions as a foundation of day-to-day risk measurement.

However, the traditional way of measuring and managing risk has been challenged by the current trading environment. Over the last several years the speed of trading has been constantly increasing. Day-trading, once the exclusive territory of floor-traders is now available to all investors. "High frequency finance hedge funds" have emerged as a new and successful category of hedge funds. Consequently, risk management is now obliged to keep pace with the market. For day traders, market makers or other very active agents on the market, risk should be evaluated on shorter than daily time intervals since the horizon of their investments is generally less than a day. For example, day-traders liquidate any open positions at closing, in order to pre-empt any adverse overnight moves resulting in large gap openings. Brokers must also be able to calculate trading limits as fast as the clients place their orders. Significant intraday variations in asset prices affect the margins a client has to deposit with a clearing firm and this should be taken into account in the design of any margining engine. Moreover, as noted by Gourieroux and Jasiak (1997), banks also use intraday risk analysis for internal control of their trading desk. For example, a trader could be asked at 11 a.m. to give his IVaR for the rest of the day.

Over recent years, most exchanges have set up low-cost intraday databases, thus facilitating access to a new type of financial information where all transactions are recorded according to their time-stamp and market characteristics (price, volume, etc.). Denoted ultra-high-frequency data in the literature (Engle, 2000), these transaction or tick-by-tick data represent the limit case for recording events (transactions, quotes, etc.): one by one and as they occur. Since the time between consecutive events (defined as duration) is no longer constant, standard time series analysis techniques are inadequate when applied directly to these transaction data.

Research on high-frequency data has progressed rapidly since Engle and Russell (1998) introduced the Autoregressive Conditional Duration (ACD) model to take into account the irregular spacing of such data. Despite a greater interest in searching for appropriate econometric models, in testing market microstructure theories, and in estimating volatility more precisely,[51] very few contributions link high-frequency data to risk management. Much effort has been spent on developing more and more sophisticated Value at Risk (VaR) models for daily data and/or longer horizons but, to the best of our knowledge, the benefit of using tick-by-tick data for market risk measures has not been sufficiently explored.

With increased access to intraday financial databases and advanced computing power, an important question arises concerning the utility of high frequency data for market risk measurement: How is one to define practical IVaR measures for investors or market makers operating on an intraday basis? The irregular

[51] Research initiated by Andersen *et al.* (2000, 2001a,b,c) on realized volatility measures (defined as the summation of high-frequency intraday squared returns) has shown dramatic improvements in both measuring and forecasting volatility.

feature of high-frequency data also makes it hard to asses the results obtained from models based on such data. The problem is further complicated by certain microstructure effects at the intraday level, effects such as the bid-ask bounce or the discreteness of prices. Giot (2002) estimates a conditional parametric VaR using ARCH models with equidistantly time-spaced returns re-sampled from irregularly time-spaced data. He also applies high-frequency duration models (log-ACD) to price durations,[52] but the results from the models for irregularly time-spaced data are not completely satisfactory.

In this paper we investigate the use of high-frequency data in risk management and we introduce an IVaR at different horizons based on tick-by-tick data. We use the Monte Carlo simulation approach for estimating and evaluating the IVaR.[53] We also propose an extension of GARCH models for tick-by-tick data (in particular, the ultra-high-frequency (UHF) GARCH model introduced by Engle, 2000) to specify the joint density of the marked-point process of durations and high-frequency returns. The advantage of this model is that it explicitly accounts for the irregular time-spacing of the data by considering durations when modelling returns. Our specification of the UHF-GARCH model is, however, more flexible than that of Engle (2000). Instead of considering returns per square root of time, we model returns divided by a nonlinear function of durations, thus endogenizing the definition of time units and letting the data speak for themselves.

A by-product of this study is represented by the out-of-sample evaluation of the predictive abilities of the UHF–GARCH model in a risk management framework. Up to now the approach taken for computing VaR with intraday data has consisted in aggregating the irregularly time-spaced data, in order to retrieve a regular sample to which traditional methods are applied.[54] This approach not only

[52] As first noted by Engle and Russell (1998), price durations (the minimum amount of time needed for the price of an asset to change by a certain amount) are closely linked to the instantaneous volatility of the price process.

[53] See Christoffersen (2003) and Burns (2002) for computing VaR using GARCH models and daily data.

[54] See for example Beltratti and Morana (1999).

requires finding the optimal aggregating scheme, but also inevitably leads to the loss of important information contained in the intervals between transactions. We propose an alternative approach which is more closely related to the literature on market microstructure and which deals directly with irregularly time-spaced data by using models adapted to the particularities of such data. From a statistical point of view, the use of more data (all the available observations in this case) and of their specific dynamics would be expected to improve the estimation of a model.

Our results show that the UHF-GARCH model performs well out-of-sample when evaluated in a VaR framework for almost all the time horizons and the confidence levels considered, even in the case when normality is assumed for the distribution of the error term, at least in our application.

The rest of the paper is organized as follows. In Section 3.2, we introduce the general econometric model used to describe the dynamics of durations and returns. In Section 3.3, we discuss the concept of IVaR and present the intraday Monte Carlo simulation approach to estimate it. An extension of Engle's (2000) UHF-GARCH model is proposed for modelling the tick-by-tick returns. In Section 3.4, we apply our methodology to transaction data on two Canadian stocks traded on the Toronto Stock Exchange (TSE): the Royal Bank of Canada (RY) stock and the Placer Dome (PDG) stock. Finally, Section 3.5 concludes and gives some possible research directions.

3.2 The general econometric model

Asymmetric-information models from the market microstructure literature suggest that both time between trades and price dynamics are related to the existence of new information on the market and that time is not exogenous to the price process. In Easley and O'Hara (1992), informed traders trade only when there are information events that influence the asset price. Therefore, in this model, long intertrade durations are associated with no news and, consequently, with low volatility. Opposite relations between duration and volatility follow from the Admati-and-Pfleiderer model (1988) where frequent trading is associated with liquidity traders. Hence, low trading means that liquidity traders are inactive, leaving a high proportion of informed traders on the market, which translates into higher volatility.

Predictions from theoretical models are relatively ambiguous. As pointed out by O'Hara, "the importance of time is ultimately an empirical question ..." (1995, p. 177). In the end, the common viewpoint is that durations and volatility are closely linked. Consequently, to estimate an IVaR that draws on insights from this microstructure literature, we use a joint modelling of arrival times and price changes.

To account for the irregularity of durations between consecutive trades, the data is statistically viewed as a marked-point process. The arrival times form the points, and random variables such as price, volume, and the bid-ask spread form the marks.

Consider two consecutive trades that happened at times t_{i-1} and t_i, at prices p_{i-1} and p_i, respectively. Let $x_i = t_i - t_{i-1}$ be the duration between these trades, and $r_i = \log(p_i/p_{i-1})$ be the corresponding continuously compounded return. Following Engle's framework (2000), a realization of the process of interest is fully described by the sequence:

$$\{(x_i, r_i), i = 1, ..., n\}.$$

The ith observation has joint density, conditional on the past filtration $\mathcal{F}_{i-1}$ given by

$$(x_i, r_i) \mid \mathcal{F}_{i-1} \sim f\left(x_i, r_i \mid \breve{x}_{i-1}, \breve{r}_{i-1}; \theta\right), \tag{3.1}$$

where θ is a d-vector of parameters ($d < n$), finite and invariant over events, and $\breve{x}_{i-1}$ and $\breve{r}_{i-1}$ denote the past of the variables X and R, respectively, up to the

$(i-1)$th transaction. The information set available at time t_{i-1} is represented by $\mathcal{F}_{i-1}$.

The joint density in (3.1) can be written as the product of the marginal density of the durations and the conditional density of the returns given the durations, all conditioned upon the past of durations and returns:

$$f\left(x_i, r_i|\breve{x}_{i-1}, \breve{r}_{i-1}; \theta\right) = g(x_i|\breve{x}_{i-1}, \breve{r}_{i-1}; \theta_1)q(r_i|x_i, \breve{x}_{i-1}, \breve{r}_{i-1}; \theta_2), \tag{3.2}$$

where $g(x_i|\breve{x}_{i-1}, \breve{r}_{i-1}; \theta_1)$ is the marginal density of the duration x_i with parameter θ_1, conditional on past durations and returns, and $q(r_i|x_i, \breve{x}_{i-1}, \breve{r}_{i-1}; \theta_2)$ is the conditional density of the return r_i with parameter θ_2, and conditional on past durations and returns as well as the contemporaneous duration x_i. Using (3.2), the log likelihood can be written as

$$\mathcal{L}(\theta_1, \theta_2) = \sum_{i=1}^{n} \left[\log g(x_i|\breve{x}_{i-1}, \breve{r}_{i-1}; \theta_1) + \log q(r_i|x_i, \breve{x}_{i-1}, \breve{r}_{i-1}; \theta_2)\right]. \tag{3.3}$$

Assuming that θ_1 and θ_2 are variation free in the sense of Engle, Hendry, and Richard (1983),[55] durations could be treated as being weakly exogenous and the two parts of the likelihood function could be maximized separately. However, the evidence suggests that this assumption is not always valid. In fact, Dolado, Rodriguez-Poo, and Veredas (2004) reject the null hypothesis of weak exogeneity for seven out of the ten models considered when analyzing high-frequency data for five stocks traded on the NYSE. Here, we estimate the two parts of the log-likelihood function simultaneously.

We model the dynamics of trade durations using the log-ACD model proposed by Bauwens and Giot (2000) (presented in the next subsection) and the price dynamics by extending Engle's UHF-GARCH framework (2000) as shown in Section 3.3.

[55] That is, if $\theta_1 \in \Theta_1$ and $\theta_2 \in \Theta_2$, then $(\theta_1, \theta_2) \in \Theta_1 \times \Theta_2$.

3.2.1 The ACD and the log-ACD models

Engle and Russell (1998) introduced the ACD model as a counterpart of the GARCH model to describe the duration clustering observed in high-frequency financial data. Since then, a plethora of models has emerged,[56] strongly supported theoretically by the recent market microstructure literature's emphasis on the information contained in the time between market events.[57]

Let ψ_i be the conditional duration given by

$$\psi_i = E(x_i|\mathcal{F}_{i-1}) \tag{3.4}$$

and supposed to be a nonnegative measurable function of $\mathcal{F}_{i-1}$.

The ACD model is based on the assumption that all the temporal dependence between durations is captured by the conditional duration mean function, such that x_i/ψ_i is independent and identically (i.i.d.) distributed,

$$x_i = \psi_i \varepsilon_i, \tag{3.5}$$

where the process $\varepsilon = \{\varepsilon_i, i \in \mathbb{Z}\}$ is a strong white noise, that is ε is an $i.i.d.$ process. We assume ε_i to be non-negative with probability one and admit a density $p(\varepsilon)$ such that $E(\varepsilon_i) = 1$.

By choosing different specifications for the conditional durations in (3.4) and the density $p(\varepsilon)$, several forms of the ACD model can be obtained. A popular one is the ACD (m, q) which is based on a linear parameterization of (3.4):

$$\psi_i = \omega + \sum_{j=1}^{m} \alpha_j x_{i-j} + \sum_{j=1}^{q} \beta_j \psi_{i-j}, \tag{3.6}$$

where $\omega > 0$, $\alpha_j \geq 0$, $j = 1, ..., m$ and $\beta_j \geq 0$, $j = 1, ..., q$, which are sufficient

[56] See Bauwens and Giot (2001), and Hautsch (2004) for extensive surveys of the existing models.

[57] See Diamond and Verrecchia (1987), Admati and Pfleiderer (1988), Easley, and O'Hara (1992) among others.

conditions to ensure the positivity of the conditional durations.

Several choices can be made for the density $p(\varepsilon)$. Engle and Russell (1998) used the standard exponential distribution (that is the shape parameter is equal to one) and the standardized Weibull distribution with shape parameter equal to γ and scale parameter equal to one. The resulting models are called the Exponential ACD (EACD) and the Weibull ACD (WACD), respectively. The choice of the distribution for the error term affects the conditional intensity function. The exponential specification implies a flat conditional hazard function which is quite restrictive and easily rejected in empirical financial applications (see e.g. Engle and Russell, 1998). The Weibull distribution (that reduces to the exponential distribution if γ equals 1) allows for a monotonic hazard function (which is either increasing if $\gamma > 1$, or decreasing if $\gamma < 1$). For greater flexibility, Grammig and Maurer (2000) advocate the use of a Burr distribution leading to the Burr ACD model, while Lunde (1999) proposes the generalized gamma distribution. Both distributions allow for hump-shaped hazard functions to describe situations where, for small durations, the hazard function is increasing and, for long durations, the hazard function is decreasing. Once the density function $p(\varepsilon)$ is specified, the parameters of the ACD model are estimated using the maximum likelihood estimation technique. In this paper we make use of the generalized gamma distribution with unit expectation which leads to the so-called GACD model.

In order to avoid the necessity of imposing non-negativity constraints on the parameters of the conditional mean function, Bauwens and Giot (2000) introduce a logarithmic version of the ACD model, called the log-ACD model, that specifies the autoregressive equation on the logarithm of ψ_i:

$$\psi_i = \exp\left(\omega + \sum_{j=1}^{m} \alpha_j \varepsilon_{i-j} + \sum_{j=1}^{q} \beta_j \ln \psi_{i-j}\right). \tag{3.7}$$

Equation (3.7) is sometimes referred to as a Nelson type ACD model because of its similarity with Nelson's EGARCH model (1991). The same hypotheses are made about ε as in the ACD model and the same probability distribution functions can be chosen. If ε_i follows a generalized gamma distribution with parameters γ_1, γ_2 ($\gamma_1 > 0$ and $\gamma_2 > 0$), its density function is given by

$$p(\varepsilon|\gamma_1, \gamma_2) = \begin{cases} \frac{\gamma_1 \varepsilon^{\gamma_1 \gamma_2 - 1}}{\gamma_3^{\gamma_1 \gamma_2} \Gamma(\gamma_2)} \exp\left[-\left(\frac{\varepsilon}{\gamma_3}\right)^{\gamma_1}\right], & \varepsilon > 0, \\ 0 \quad , \text{ elsewhere.} \end{cases}$$

where $\Gamma(\cdot)$ denotes the gamma function and $\gamma_3 = \Gamma(\gamma_2) / \Gamma\left(\gamma_2 + \frac{1}{\gamma_1}\right)$. The generalized gamma distribution nests the Weibull distribution when $\gamma_2 = 1$. Estimation is performed in the same way as for the ACD model by considering the new specification of ψ_i.

3.2.2 The UHF-GARCH model

Once we model the durations between trades conditional on past information, we need to specify a model for price changes, conditional on the current duration and past information. Since high-frequency data are irregularly time-spaced, Engle (2000) proposes a GARCH model that takes into consideration this feature of tick-by-tick data. Let the conditional variance per transaction be

$$h_i = V_{i-1}(r_i | x_i), \tag{3.8}$$

where the conditioning information set contains the current durations as well as the past returns and durations. However, as argued by Engle (2000) and Meddahi, Renault and Werker (2003), the variance of interest is not the "traditional" variance but the variance per unit of time, naturally defined as:

$$V_{i-1}\left(\frac{r_i}{\sqrt{x_i}} | x_i\right) = \sigma_i^2, \tag{3.9}$$

which naturally implies that $h_i = x_i \sigma_i^2$.

Under the assumption $E_{i-1}(r_i | x_i) = 0$, the volatility per unit of time is then modeled as a simple GARCH(1,1) process:

$$\sigma_i^2 = \widetilde{\omega} + \widetilde{\alpha}\left(\frac{r_{i-1}}{\sqrt{x_{i-1}}}\right)^2 + \widetilde{\beta}\sigma_{i-1}^2, \tag{3.10}$$

where $\widetilde{\omega} > 0$, $\widetilde{\alpha} \geqslant 0$ and $\widetilde{\beta} \geqslant 0$.

Although it seems natural to model the variance as a function of time when using irregularly time-spaced data, the above modelling for the unit of time appears as quite restrictive, since the conditional heteroskedasticity in the returns could depend on time in a more complicated way. In the next section, we shall propose a useful extension of the above UHF-GARCH model that is more flexible and seems to provide a better adjustment of the data, at least in our empirical applications.

3.3 Monte Carlo IVaR

3.3.1 IVaR: definition

To define our IVaR, let $Y = \{y_k, k \in \mathbb{Z}\}$ be the process of the asset return re-sampled at regular time intervals equal to T units of time. We consider a realization of length n' of the process Y $\{y_k; k = 1, ...n'\}$ with y_k obtained at times t'_k such that $t'_k - t'_{k-1} = T$. Thus, y_k denotes the T-period return which is simply the sum of correspondent tick-by-tick returns

$$y_k = \sum_{i=1}^{\tau(k)-1} r_i, \tag{3.11}$$

where $\tau(k)$ is such that the cumulative duration exceeds T for the first time. This means that

$$\sum_{i=1}^{\tau-1} x_i \leq T \text{ and } \sum_{i=1}^{\tau} x_i > T.$$

Thus, by modelling the durations specifically with the ACD model, we are able to determine every $\tau(k)$ and keep track of the time step.

The IVaR with confidence level $1-\alpha$, is formally defined as

$$\Pr(y_k < -IVaR_k(\alpha) \mid \mathcal{G}_k) = \alpha. \tag{3.12}$$

where $\mathcal{G}_k$ denotes the information set that includes all the tick-by-tick data (durations and returns) up to time t'_k . Common confidence levels used in the applications are $1-\alpha = 95\%, 97.5\%, 99\%$ and 99.5%; we shall consider these values in the empirical study. It follows from (3.12) that the IVaR is such that $-IVaR_k(\alpha) = Q_k(\alpha \mid \mathcal{G}_k)$ where $Q_k(\alpha \mid \mathcal{G}_k)$ is the quantile of the conditional distribution of y_k.

Various methods exist for computing a VaR: parametric or variance-covariance methods (e.g. RiskMetrics, GARCH), nonparametric (e.g. Historical Simulation), and semi-parametric (e.g. CAViar, extreme value theory). We propose a GARCH-type model, as presented in the next subsection, to specify the time-varying volatility of the tick-by-tick returns r_i and a Monte Carlo simulation approach to simulate the future regularly spaced returns y_k from which the IVaR is computed.

3.3.2 The extended UHF-GARCH model

In this subsection, we propose a more flexible specification of the UHF-GARCH model presented in Section 3.2.2 in which the time weighting is determined endogenously. We model the volatility of returns as the product of a function of duration and a GARCH component. More precisely, let

$$\frac{V_{i-1}\left(r_i|x_i\right)}{x_i^{\gamma}} = \sigma_i^2, \tag{3.13}$$

which implies that

$$h_i = x_i^{\gamma}\sigma_i^2. \tag{3.14}$$

The parameter γ that specifies the duration weighting for the volatility of a particular stock has to be estimated. This formulation allows us to specify a more general form of heteroskedasticity in the conditional variance of the returns.

In Section 3.4, we use transaction prices instead of mid-quotes for forecasting volatility in an order-driven market. Transaction prices may be affected by various market microstructure effects, such as nonsynchronous trading and the bid-ask bounce (Tsay, 2002), that may induce serial correlation in high-frequency returns. Therefore, to remove microstructure effects, we follow Grammig and Wellner (2002) and Ghysels and Jasiak (1998) and use an ARMA(1,1) model on the tick-by-tick returns:

$$r_i = c + \varphi_1 r_{i-1} + e_i + \theta_1 e_{i-1}. \tag{3.15}$$

We model the GARCH component as

$$\log \sigma_i^2 = \widetilde{\omega} + \sum_{j=1}^{P} \widetilde{\beta}_j \log \sigma_{i-j}^2 + \sum_{j=1}^{Q} a_j \left\{ \frac{|e_{i-j}|}{h_{i-j}^{1/2}} - E\left(\frac{|e_{i-j}|}{h_{i-j}^{1/2}} \right) \right\} + \sum_{j=1}^{Q} \widetilde{\alpha}_j \left(\frac{e_{i-j}}{h_{i-j}^{1/2}} \right), \tag{3.16}$$

where $e_i = z_i\sqrt{h_i}$ and $z = \{z_i, i \in \mathbb{Z}\}$ denotes a strong white noise, that is the z_i's are $i.i.d.$ with a zero mean and unit variance. The unknown parameters are c, φ_1, θ_1, $\widetilde{\omega}$, $\widetilde{\beta}_j$, a_j, $\widetilde{\alpha}_j$ and γ. When $\gamma = 1$ the model is similar, though not equivalent, to Engle's UHF-GARCH model (2000) represented by equations (3.8), (3.9) and (3.10). We specify the mean equation (3.15) on the returns r_i rather than on the returns weighted by a function of durations, because we want to simulate the future distribution of returns from which the IVaR could be extracted. As shown in the empirical part of the article, the estimate of parameter γ is far below unity and is statistically significant, at least in our empirical application. When $\gamma = 0$, the model becomes a standard GARCH applied to the high-frequency data by ignoring the irregular spacing of returns when modelling the volatility. The use of an EGARCH*(p,q)* model is justified by the advantage of keeping the volatility component positive during simulations regardless of the variables added to the autoregressive equation. The absence of non-negativity constraints on the parameters also simplifies numerical optimization.

3.3.3 Intraday Monte Carlo simulation

In this subsection, we describe the Monte Carlo simulation approach used for computing the IVaR based on tick-by-tick data. As the time intervals between consecutive observations are irregular, the question about how to assess the results needs to be considered. Since the model we use fully specifies the distribution of returns in event time, one-step forecasts can be computed analytically. However, we need to run simulations in order to obtain forecasts of returns in any arbitrary length of calendar time, T. More specifically, the ACD model will define the time step of our simulations and the extended UHF-GARCH model introduced in the previous subsection will generate the corresponding tick-by-tick returns. We compute the forecasted returns for regular calendar-time intervals as the sum of irregular (tick-by-tick) intraday returns. Using the simulated distribution of returns over the chosen time interval T, we calculate the IVaR by extracting the desired percentile. The IVaR obtained is thus an IVaR for regular time intervals (therefore, comfortably comparable to regular real returns), but computed using tick-by-tick data and adapted to the non-regularity of time intervals.

More precisely, we proceed as follows:

(1) The original sample is divided into two parts, one for estimation and one for forecast/validation. The estimation sub-sample serves to calibrate a log-ACD-GARCH model for durations and tick-by-tick returns as presented in the previous sections.

(2) We draw random numbers from the standard normal distribution and the standard generalized gamma distribution, respectively, to compute the innovations which will serve to generate scenarios for future durations, returns, and volatilities.

(3) In order to obtain the simulated durations within the first fixed interval, we forecast the durations in an iterative way using equations (3.5) and (3.7) as long as the sum of forecasted tick-by-tick durations does not exceed the time interval of interest.

(4) Using equations (3.14), (3.15), and (3.16), we forecast, conditional on the simulated durations, the tick-by-tick returns corresponding to this first regular interval. By summing up all these returns we obtain the (regular) return for the first interval.

(5) We continue the procedure until the desired number of fixed intervals or regular returns is obtained. This corresponds to the first path.

(6) We repeat steps (1) to (5) for the desired number of paths.

(7) For each time-fixed interval, the IVaR corresponds to the quantile of the simulated distribution of returns for that interval.

(Insert Figure 3.1 here)

Figure (3.1) resumes the main steps of the simulation. As one can see, the algorithm displays similarities with the traditional Monte Carlo simulation used for computing a multi-period (generally 10-day) VaR. The main difference is at step (3) where we make use of the ACD model to keep trace of the time step. While, in the standard Monte Carlo approach the time unit is not very important because all observations are equally spaced, here we relax this constraint and are able to proceed tick-by-tick by using models adapted to the irregularity of time intervals, such as the ACD and the UHF-GARCH models. We illustrate this procedure using real data in the next section.

3.4 Empirical study

3.4.1 Data

In this section we compute the IVaR according to the methodology previously described using high-frequency data from the "Market Data Equity Trades and

Quotes Files" CD-ROM of the Toronto Stock Exchange (TSE). To our knowledge, this is the first econometric study analyzing (irregular) high-frequency data from the Canadian stock market. Previous studies used data from the NYSE (e.g., Engle and Russell 1998; Giot 2002), or from the Paris Bourse (e.g., Jasiak 1998; Gouriéroux, Jasiak and Le Fol, 1999), the German Stock Exchange (e.g., Grammig and Wellner 2002), and the Moscow Interbank Currency Exchange (Anatolyev and Shakin 2004). Thus, this study adds another dimension to previous work. We focus on the Royal Bank of Canada stock (RY) and the Placer Dome stock (PDG) for the period from April 1st to June 30, 2001 which contains 63 trading days. The Royal Bank is Canada's largest bank as measured by market capitalization (about US$ 33.4 billion) and assets. Placer Dome is Canada's second largest gold miner with a market capitalization of US$ 8.2 billion at the end of 2004. Both stocks belong to the Toronto 35 Index. The Toronto 35 Index was developed by the TSE in 1987 and consists of the 35 largest and most liquid stocks in Canada. As of April 30, 2001 the relative weights of the RY stock and PDG stock in the Toronto 35 Index were 4.44% and 1.15%, respectively.

In 1999, the Toronto Stock Exchange (TSE) became Canada's sole exchange for the trading of senior equities. The TSE is the seventh largest equities market in the world as measured by domestic market capitalization, with US$ 1,178 billion at the end of 2004 (source: World Federation of Exchanges Annual Report 2004). The TSE currently trades equities for approximately 1,485 listed firms, with a daily trading volume averaging CAD$ 3,3 billion in 2004. The TSE operates as an automated, continuous auction market. Limit orders enter the queues of the order book and are matched according to a price-time priority rule. There is a pre-opening session from 7:00 to 9:30 am during which market participants can submit orders for possible execution at the beginning of the regular session. The pre-opening session involves the determination of a Calculated Opening Price (COP) that equals the price at which the greatest volume of trades can trade or, if it is not unique, the price at which there is the least imbalance or the price closest to the previous closing price (see Davies, 2003 for details about the pre-opening session at the TSE and the role of the registered trader at the pre-opening). Regular trading starts at 9:30 and ends at 16:00. Opening trades are at the COP and, as they have a different dynamic we eliminate them from the empirical analysis . Each stock is assigned to a Registered Trader (RT) who is required to act as a market maker and to maintain a fair, orderly, and continuous two-sided market for that stock. The RT contributes to market liquidity and depth and reduces volatility by buying or selling against the market. He must also guarantee all oddlot trades and trade all orders of a certain size, known as the Minimum Guaranteed Fill (MGF) orders, within a set price difference between buy and sell orders. The RT resembles the specialist at the NYSE but he does not act as an agent for client order flow and does not have exclusive knowledge of the limit order book. Unlike the NYSE, trading on the TSE is completely electronic, without any floor trading. An order book open to subscribers insures a highly transparent market.[58]

The TSE intraday database contains date-and-time stamped bid-and-ask quotes, transaction prices, and volume for all firms. Special codes identify special trading conditions. Prior to the analysis, a couple of operations need to be conducted on the data. Following the literature, we removed all non-valid trades and interdaily durations and kept only those transactions made during regular trading hours. As already mentioned, open trades are also deleted in order to avoid effects induced by the opening auction. For simultaneous transactions, we consider a weighted average price and remove all remaining observations with this time stamp, thus considering these observations as split transactions.

Previous studies dealing with high-frequency data commonly used mid-quotes instead of transaction prices (e.g., Engle 2000, Engle and Lange 2001, Manganelli 2002). While this may be appropriate for the NYSE which is a quote-driven market, working with transaction prices appears a better choice in our case since we are interested in VaR estimation and want to forecast returns for real

[58]The future of traditional floor trading at the NYSE was questioned after the merger on April 20, 2005 of the NYSE with the electronic trading network Archipelago.

transactions. To liquidate positions in an order-driven market one has to transact either on the ask or the bid and therefore using the midquote-change quantiles may understate the true VaR.[59]

Extremely large durations and returns between two successive trades are very unlikely, therefore we filter out all the observations with absolute returns larger than 10 standard deviations and durations larger than 25 standard deviations, resulting in 2 outliers for the RY stock and 9 outliers for the PDG stock. These leaves us with a total of 51,660 observations for the RY stock and 27,956 observations for the PDG stock.

(Insert Figure 3.2 here)

Figure (3.2) displays the histograms of the transaction returns for the two stocks considered. The price increment for the two stocks RY and PDG for the period analyzed was one penny (C$0.01)[60] We observe a disproportionately large number of zero returns (almost 60%) which seems rather typical for high-frequency data, especially for single stocks. For the IBM dataset used by Engle and Russell (1998), Tsay (2002, p.182) reported that about two-thirds of the intraday transactions were without price change. Gorski, Drozdz and Speht (2002) report the same phenomenon for DAX[61] returns and call it the *zero return enhancement*. Bertram (2004) also finds a large number of zero price changes for equity data from the Australian Stock Exchange and argues that zero returns follow from the absence of significant new information on the market. According to the efficient market hypothesis, a price changes when new information arrives on the market and, consequently, traders simply continue to trade at the previous price when the amount of information is insufficient to move the price.

[59] We are grateful to Joachim Grammig for shedding light on this issue.

[60] Starting on January 29, 2001 the TSE introduced the penny tick size for stocks selling at over $0.50.

[61] Deutsche Aktienindex (DAX) represents the index for the 30 largest German companies quoted on the Frankfurt Stock Exchange.

Information about the raw data is given in Table (3.1). Of the two stocks, RY with an average duration of almost 29 seconds is traded almost twice as frequently as PDG, while PDG is traded on average every minute. We also note overdispersion of durations, i.e., the standard deviation is higher than the mean. This is typically found in the literature for the trade duration process and it may suggest that the exponential distribution cannot properly describe the durations. The transaction returns have a sample mean equal to zero for both stocks and a standard deviation equal to 0.001 and 0.002, respectively. RY exhibits positive sample skewness while PDG has a negative skewness. Both stocks' returns display a kurtosis higher than that of a normal distribution.

(Insert Table 3.1 here)

3.4.2 Seasonal adjustment

As noted by Engle and Russell (1998), high-frequency data exhibits a strong intraday seasonality explained by the fact that markets tend to be more active at the beginning and towards the end of the day. While most of the studies on high-frequency data ignore interday variations in variables, Anatolyev and Shakin (2004) found that durations and return volatilities of the Russian stocks considered fluctuate throughout different trading days. To prevent the distortion of results, these interday and intraday seasonalities must be taken out prior to the estimation of any model . We inspected our data for such evidence. We noted for example that durations are higher on Mondays and Fridays than during the rest of the week. A Wald test in a regression of average RY durations on five day-of-the-week dummies rejects the null hypothesis of equality of all coefficients, thus providing evidence of interday seasonal effects. No interday effects are identified for PDG durations and returns. When found, we remove interday seasonality under a multiplicative form, following the approach taken by

Anatolyev and Shakin (2004):

$$x_{t,inter} = \frac{x_t}{\overline{x}_s}, \tag{3.17}$$

$$r_{t,inter} = \frac{r_t}{\sqrt{\overline{r^2}_s}}, \tag{3.18}$$

where $\overline{x}_s$ corresponds to the average duration for day s if observation t belongs to day s, and $\overline{r^2}_s$ is the average of squared returns for day s.

To take out the time-of-day effect, Engle and Russell (1998) suggest computing "diurnally adjusted" durations by dividing the raw durations by a seasonal deterministic factor related to the time at which the duration was recorded. We obtain intraday seasonally adjusted (isa) durations and returns in the following way:

$$x_{t,intra} = \frac{x_{t,inter}}{E(x_{t,inter}|\mathcal{F}_{t-1})},$$

$$r_{t,intra} = \frac{r_{t,inter}}{\sqrt{E(r^2_{t,inter}|\mathcal{F}_{t-1})}}.$$

where the expectation is computed by averaging the variables over thirty-minute intervals for each day of the week and then using cubic splines on the thirty-minute intervals to smooth the seasonal factor. Thus, intraday patterns are different for different days of the week. Figures (3.3) and (3.4) show, respectively, the estimated intraday factor for durations and squared returns for the RY stock. Similar seasonal-factors patterns are found for the PDG stock and therefore they are not reported. The patterns are analogous to what has been found in previous studies. Durations are shorter at the beginning and at the end of the day and longer in the middle. The return volatility is lower in the middle of the day than at the beginning and at the end. This reflects the behavior of traders who are very active at the beginning of the trading session and adjust their positions to incorporate the overnight change in information. Towards the end of the day, traders are changing their positions in anticipation of the close and to pre-empt the risk posed by any information that could arrive during the night. We also notice a difference between the patterns for different days.

(Insert Figures 3.3 and 3.4 here)

Descriptive statistics for deseasonalised data are given in Table (3.2). We notice the salient features of high-frequency data, such as overdispersion in trade durations and high autocorrelations. The Ljung-Box statistics for fifteen lags Q(15) tests the null hypothesis that the first 15 autocorrelations are zero. The large values of these statistics greatly exceed the 5% critical value of 25, indicating strong autocorrelation of both durations and returns. The skewness is close to zero for both stocks. There is still excess kurtosis for both stocks even if at a lesser degree compared to the raw data. We chose the normal distribution when estimating the GARCH model and we obtained satisfactory simulation results. It is well known that the conditional normality assumption in ARCH models generates some degree of unconditional excess kurtosis (see, e.g., Bollerslev et al., 1992).

(Insert Table 3.2 here)

3.4.3 Estimation results

In this section we apply the model presented in Sections 3.2 and 3.3 to the deseasonalised data for the two stocks. Observations of the first month are used for estimation and those of the last two months serve for forecast and validation. The likelihood function is maximized using Matlab v. 7 with the Optimization toolbox v. 3.0 and numerical derivatives are used for computing the standard errors of the estimates.

We first tested our durations for the clustering phenomenon using the test of Ljung-Box with 15 lags, Q(15). The high coefficients (reported in Table 3.2) suggest the presence of ACD effects in our durations data at any reasonable significance level. The high positive serial dependence of the squared returns greatly exceeding the critical values, as illustrated by the Ljung-Box test statistics, noted $Q_2(15)$, represents evidence of volatility clustering and justifies the application of a GARCH-type model.

Estimation results are presented in Table 3.3 together with the p-values for the adequacy tests applied to the standardized residuals. We tried to find the best fit for our data and reestimated the model chosen without the variables that were statistically insignificant. We tested the adequacy of the log-GACD model using the Ljung-Box test applied to the standardized residuals.[62] We judged the quality of the GARCH fit using the Ljung-Box tests for standardized residuals and their squares.

(Insert Table 3.3 here)

We retain a log-GACD(2,2)-ARMA(1,1)-UHF-EGARCH(1,1) model for our data. We find that a log-GACD(2,2) specification is successful in removing the autocorrelation in durations. For both stocks the p-values of the Ljung-Box test with 15 lags are superior to 0.05. The parameters of the generalized gamma distribution are all significant. For each stock, the sum of the autoregressive parameters $\beta_1 + \beta_2$ is close to one, revealing high persistence in durations.

With regard to the high-frequency returns, the ARMA(1,1)-UHF-EGARCH(1,1) specification accounts satisfactorily for the dependence of both the returns and squared returns of the PDG stock, as evidenced by the p-values of the order-15 Ljung-Box test statistics that are all greater than 0.05. For the RY stock, the Ljung-Box test statistics indicate that some serial dependence is still present in the data, which is inconsistent with the model's adequacy. However, the dependence is dramatically reduced compared to the original data. The Ljung-Box statistic with 15 lags $Q(15)$ for autocorrelation of returns was reduced from 8195 to 62 (the associated 95% critical value being 24.99). Similarly, the

[62] We have also applied the test statistics for ACD adequacy introduced by Duchesne and Pacurar (2005) using the Bartlett kernel. The results are similar and therefore not reported.

Ljung-Box statistic $Q_2(15)$ for autocorrelation of squared returns was reduced from 15417 to 36. Similar results have been found by Engle (2000) using Ljung-Box test statistics. The problem of passing tests of model adequacy seems to remain an issue when using irregular high-frequency data. We have tried higher-order models both for the mean and the variance equations but we have not been able to gain any considerable improvement. We keep the ARMA(1,1) - UHF- EGARCH(1,1) specification of the model for its parsimony and because we are rather interested in assessing its forecasting ability under a risk management framework. As in Engle (2000), the MA(1) coefficient represented by θ_1 is negative and highly significant for both stocks. The AR(1) term represented by φ_1 is positive. This can be explained by the fact that traders split large orders into smaller orders to obtain a better price overall and therefore make prices move in the same direction and thus induce a positive autocorrelation of returns (Engle and Russell, 2005). The positive autocorrelation can also be related to negative feedback trading (Sentana and Wadhwani, 1992). Engle (2000) has also found evidence of positive autocorrelation in high-frequency returns. The autoregressive parameter $\widetilde{\beta_1}$ for the EGARCH(1,1) model is close to one for the RY stock, indicating a higher persistence in volatility than for the PDG stock. The parameter $\widetilde{\alpha_1}$ is statistically insignificant (not reported in the table) for both stocks, so no leverage effect is supported. The parameter γ is approximately equal to 0.05 for both stocks and it is statistically different from zero.

Now that we have calibrated the model to our data, we may proceed to the simulation of future durations and returns.

3.4.4 IVaR backtesting

In this section we simulate tick-by-tick durations and returns using the estimated coefficients of our model and the observations of the last day as starting values. We sum up the irregular simulated returns over a fixed-time interval. Five different interval lengths are used: 15, 25, 35, 45 and 90. Since all models are applied on deseasonalised data, the time unit of the interval does not represent a calendar unit (e.g. seconds, minutes). A correspondence can nevertheless be established for each stock given the number of resampled intervals generated, depending on the trading intensity of that stock for the 44 days of the validation period. For example, a length = 15 for the RY stock results in 2470 intervals for 44 trading days, which is equivalent to an average interval of 7 minutes. Assessing the results in a regularly spaced framework allows us to evaluate the performance of the model in a traditional way, thus circumventing the problem of finding an appropriate benchmark. We have generated 5000 independent paths and we have extracted the IVaR as a percentile from the simulated distribution of returns.

We first analyze the performance of the model by using Kupiec's test (1995) for the percentage of failures, which is also embedded in the regulatory requirements on the backtesting of VaR models. Using a standard procedure in the literature, we compute the empirical failure rate ($\widehat{\alpha}$) as the percentage of times actual returns (y_k) are greater than the estimated IVaR. If the IVaR estimates are accurate, the failure rate should equal α. Kupiec's test checks whether the observed failure rate is consistent with the frequency of exceptions predicted by the IVaR model. Under the null hypothesis that the model is correct we have $\widehat{\alpha} = \alpha$ and Kupiec's likelihood ratio statistic takes the form:

$$LR = 2\left[\ln(\widehat{\alpha}^m(1-\widehat{\alpha})^{n-m}) - \ln(\alpha^m(1-\alpha)^{n-m})\right]$$

where m is the number of exceptions and n is the sample size. This likelihood ratio is asymptotically distributed as a $\chi^2(1)$ under the null hypothesis. The left panels of Tables (3.4) and (3.5) show the p-values for the Kupiec test for the two stocks with a 5%, 2.5%, 1% and 0.5% IVaR level. Bold entries denote a failure of the model at the 95% confidence level, since the p-value is inferior to 0.05. The results show that the model performs well for both stocks. For almost all the intervals and the tails considered the p-values are superior to 0.05. The only exception is for smallest interval ($T = 15$) and higher VaR levels ($\alpha = 5\%, 2.5\%$)

which could be explained by the fact that the interval length is too small for obtaining non-zero returns when resampling the tick-by-tick real returns and, therefore, the theoretical number of IVaR violations cannot be achieved. For an interval equal to 25 and a 5% IVaR level, the model is rejected at a 95% confidence level but not at a 97.5% confidence level.

We also apply the dynamic quantile (DQ) test of Engle and Manganelli (2004) to check for another property a VaR measure should display: the hits (IVaR violations) should not be serially correlated. According to Engle and Manganelli (2004), this can be tested by defining a sequence:

$$Hit_k \equiv I(y_k < -IVaR_k) - \alpha.$$

The expected value of Hit_k is 0 and the DQ test is computed using the regression of the variable Hit_k on its past, on current IVaR, and any other variables:

$$Hit_k = XB + \epsilon_k$$

Then, $DQ = \widehat{B}'X'X\widehat{B}/(\alpha(1-\alpha)) \sim \chi^2(l)$, where l is the number of explanatory variables and $\widehat{B}$ is the OLS estimate of B. We perform the test using 5 lags of the hits and the current IVaR as explanatory variables. Results are given in the right panels of Tables (3.4) and (3.5) which report the p-values of the DQ test. The results are satisfactory for the RY stock and coherent with the results from Kupiec's test. In most cases the p-values are larger than 0.05. For the PDG stock, some estimates still show some predictability, especially for the smallest interval. Overall, it seems that the model performs best for the two stocks when a 1% IVaR level is considered.

(Insert Tables 3.4 and 3.5 here)

Figure (3.5) illustrates the typical IVaR profile obtained from the model. One might argue that the Monte Carlo simulation is very time-consuming. While this may be true for backtesting purposes that generally require a sufficiently large number of validation intervals, computing the next forecasted IVaR (which is normally required for practical purposes) is reasonably fast. For example, producing the next IVaR with a Pentium 4, one CPU 3.06 GHz and 5000 simulated paths for an interval equal to 90, takes us approximately 3.5 minutes for each of the two stocks considered. For our samples, an interval equal to 90 corresponds on average to 41 minutes for RY and 79 minutes for PDG. The computing time can easily be reduced to less than one minute by using more than one CPU.

(Insert Figure 3.5 here)

3.5 Conclusion

In this paper, we have proposed a way of computing intraday Value at Risk using tick-by-tick data within the UHF-GARCH framework introduced by Engle (2000). Our specification of the UHF-GARCH model is more flexible than that of Engle (2000) since it endogenizes the definition of the time unit and lets the data speak for themselves. We applied our methodology to two actively traded stocks from the TSE (RY and PDG). While the literature is full of efforts to develop sophisticated VaR models for daily data, here we investigate the use of irregularly time-spaced intraday data for risk management. This is particularly useful for defining an IVaR appropriate for agents who are very active in the market. As a by-product of our study, we provide an out-of-sample evaluation of the predictive abilities of an UHF-GARCH model in a risk management framework, a question that has not yet been addressed in the literature.

We developed an intraday Monte Carlo simulation approach which enabled us to forecast high-frequency returns for any arbitrary interval length, thus avoiding complicated time manipulations in order to return to a convenient regularly spaced framework. In our setup the ACD model yields the consecutive steps in time while the UHF-GARCH model allows us to simulate the corresponding conditional tick-by-tick returns. Regularly spaced intraday returns are simply the sum of tick-by-tick returns simulated conditional on the forecasted duration. We considered using a normal distribution to estimate the UHF-GARCH model but, instead of using the predicted volatility given by that model and computing the VaR in a parametric way, we preferred to simulate the distribution of returns and extract the IVaR from it. Therefore, the approach is not totally model dependent.

Our results for the RY and PDG stocks indicate that the UHF-GARCH model performs well out-of-sample even when normality is assumed for the distribution of the error term, provided that the intraday seasonality has been accounted for prior to the estimation. This may be explained by the fact that we use a semiparametric method for computing the VaR as an empirical quantile from the simulated distribution of returns. In this way, it becomes possible to define an IVaR for any horizon of interest based on tick-by-tick data. Thus, our methodology for computing VaR with tick-by-tick data may constitute a reliable approach for measuring intraday risk.

Potential users of our approach would be traders who need intraday measures of risk; brokers and clearing firms looking for more accurate computations of margins; or any other entity interested in computing the VaR during a trading day in order to improve risk control. To compute the next IVaR using the method we propose, one has to monitor the time using the starting time (i.e. 11 a.m.) and the simulated durations for each path and, consequently, to apply the appropriate seasonal factors to re-introduce both interday and intraday seasonality in returns. The IVaR extracted from the simulated raw returns then has the usual interpretation.

Several extensions follow naturally from this study. First, a comparison of the VaR based on tick-by-tick data with predictions obtained from volatility models of the ARCH type (on regularly spaced observations) and realized volatility models in the spirit of Giot and Laurent (2004) would help clarifying the advantage of each approach. Second, one may wonder whether banks could benefit from incorporating tick-by-tick information into their VaR models.

Financial institutions and particularly banks currently compute the 1-day VaR based on end-of-day positions, taking into account only daily closing prices and thus ignoring possibly wide intraday fluctuations and, consequently, the risks associated with them. Therefore, one could use our approach to see whether better results can be obtained by using all the trade information available in intraday databases rather than extracting a single observation to characterize the activity of a whole day. Third, the impact of using more sophisticated UHF-GARCH models together with distributions that account for fat tails and the excessive number of zero returns could also be investigated. Fourth, given that several methods for dealing with the intraday seasonality have been proposed in recent literature, one may study the impact of different deseasonalization procedures on the IVaR estimates. Finally, a challenging but rewarding extension would be the development of a portfolio IVaR based on irregularly time-spaced high-frequency data.

One may argue that the true VaR is underestimated by using transaction data since transactions are observed only if the spread is favorable to the trader. In this respect, since transaction events are certainly timed liquidity risk is not taken into account [63]. A VaR based on knowledge of the order book such as that proposed by Giot and Grammig (2005) could provide an upper bound of the estimate.

[63] We thank Joachim Grammig for pointing it out.

3.6 References

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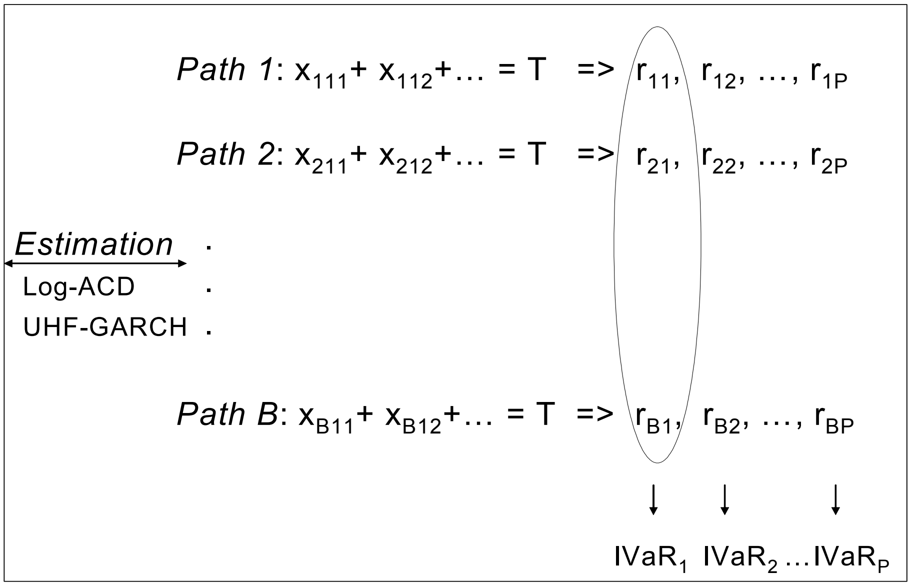
Figure 3.1: Illustration of the intraday Monte Carlo simulation approach



B is the number of independently simulated paths. T is the length of the chosen time interval. P is the number of regular intervals used for validation. x_{ijk} is the k trade duration corresponding to the path i for the interval j. r_{ij} is the regular return corresponding to the interval j for the path i. i = *1,…,B*. j = *1,…,P*. k = *1, …,* t such that $\sum_{k=1}^{t} x_{ijk} \leqslant T$ and $\sum_{k=1}^{t+1} x_{ijk} > T$.

Figure 3.2: Histograms of intraday returns for Royal Bank (RY) and Placer Dome (PDG)

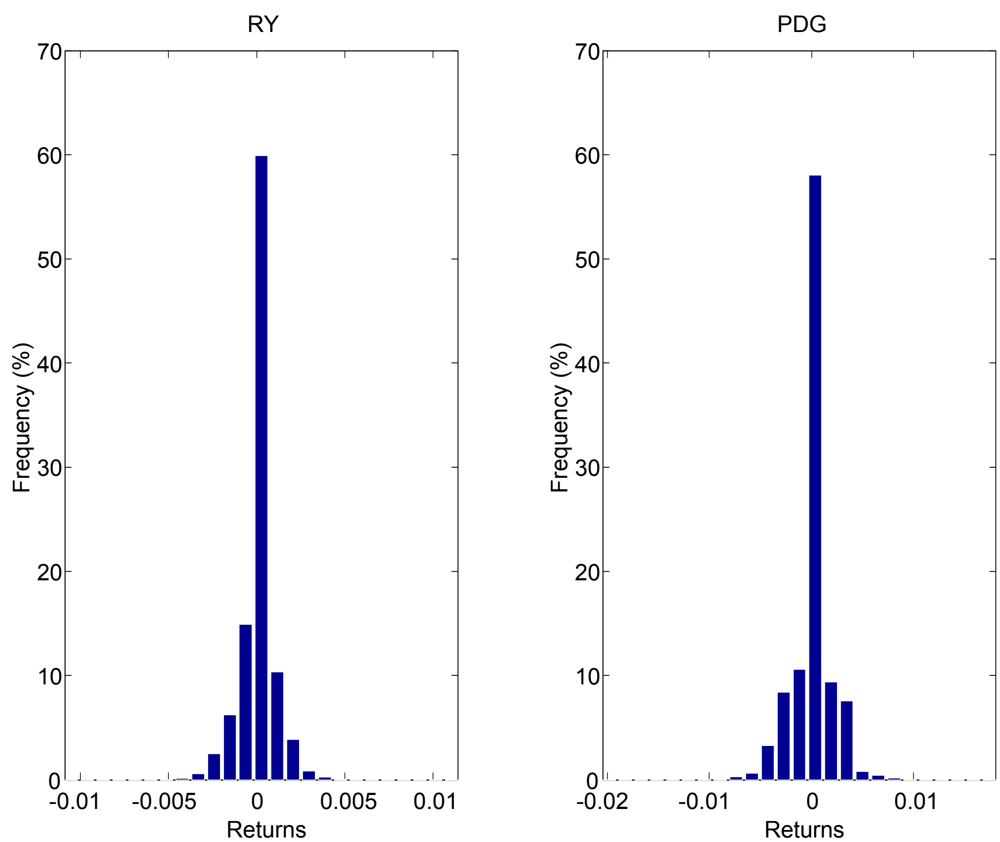


Figure 3.3: Estimated intraday factors for RY durations

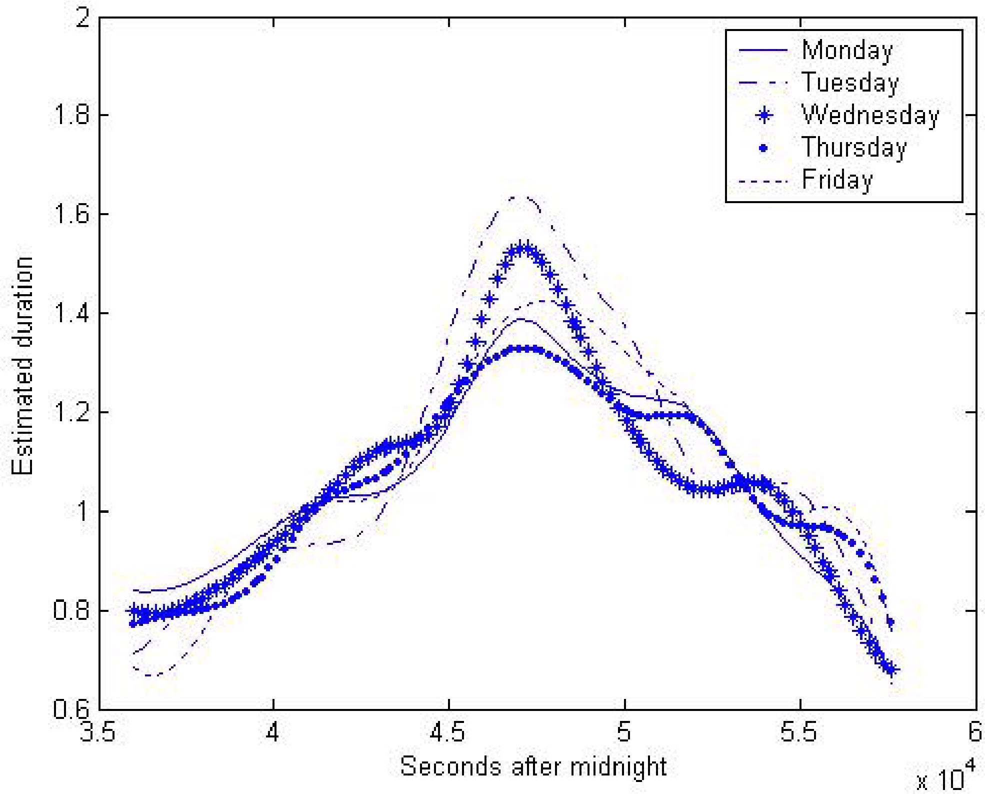


Figure 3.4: Estimated intraday factors for RY squared returns

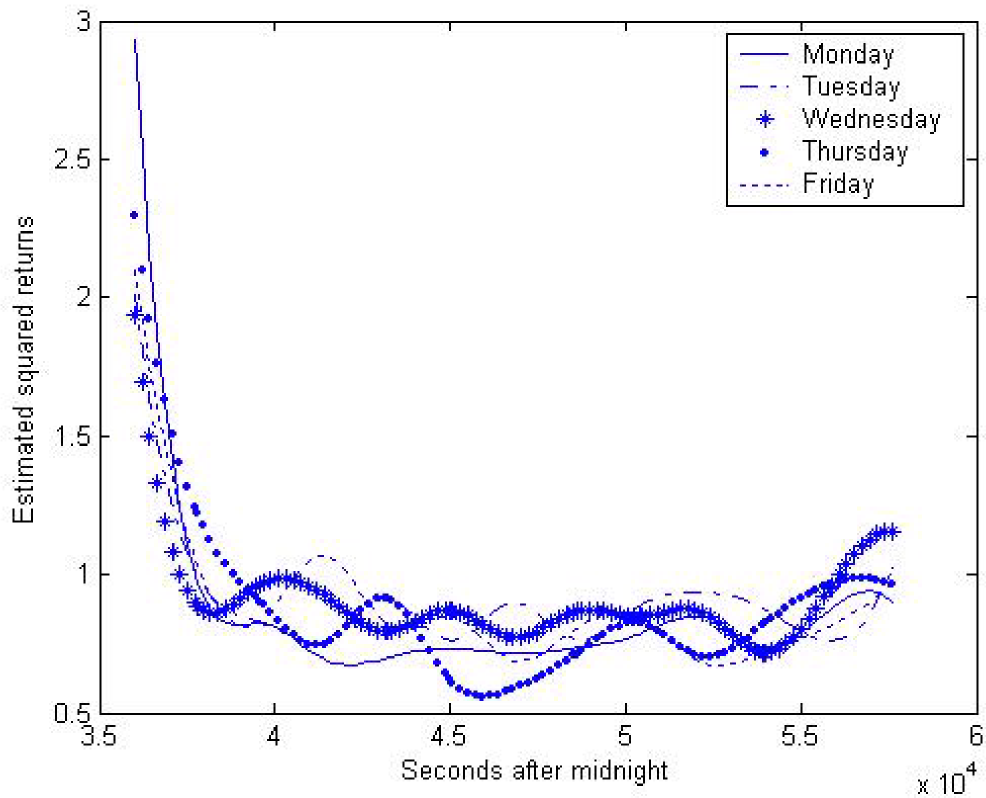


Figure 3.5: Intraday returns vs IVaR for RY (interval = 45 or 22 minutes)

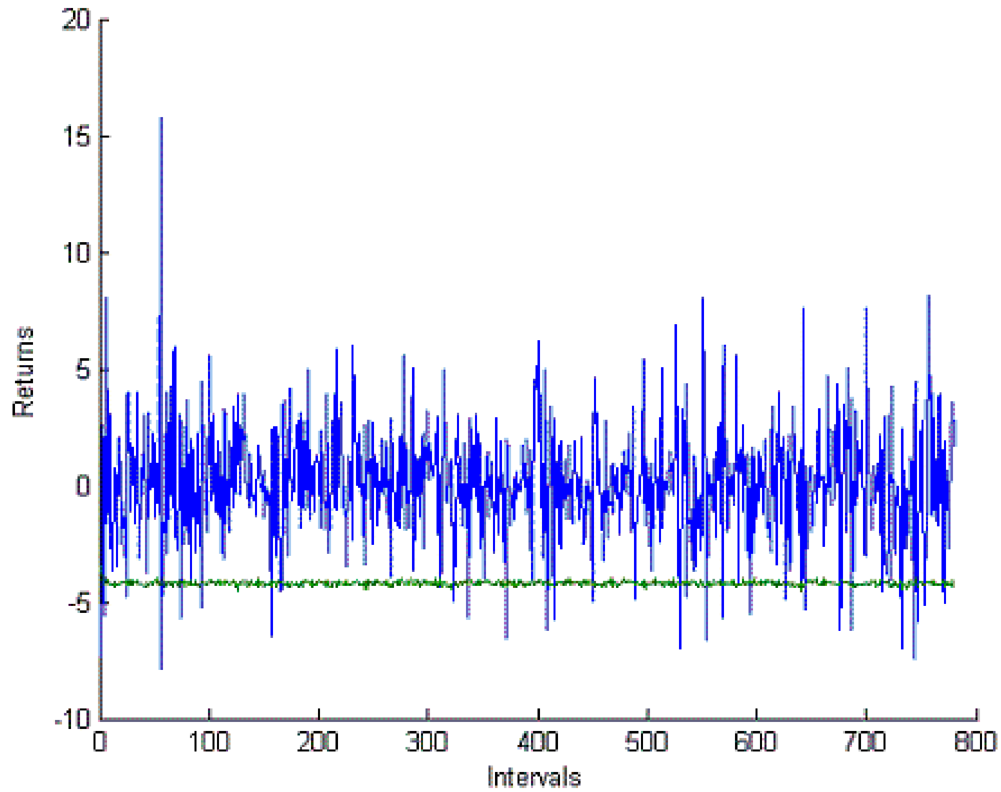


Table 3.1: Descriptive statistics of raw data for Royal Bank (RY) and Placer Dome (PDG) Stocks

	Mean	Std. dev	Skew	Kurt	Max	Min
Durations						
RY	28.44	35.19	2.98	17.54	526	1
PDG	52.59	82.11	5.04	59.87	2248	1
Returns						
RY	0.000	0.001	0.024	9.890	0.011	-0.011
PDG	0.000	0.002	-0.169	11.195	0.018	-0.020

Note: The sample period runs from April 1st to June 30, 2001. It consists of 51,660 observations for the RY stock and 27,956 observations for the PDG stock. *Mean* is the sample mean, *Std. dev* is the sample standard deviation, *Skew* is the sample skewness coefficient, *Kurt* is the sample kurtosis, *Max* is the sample maximum, *Min* is the sample minimum.

Table 3.2: Descriptive statistics of deseasonalised data for Royal Bank (RY) and Placer Dome (PDG) stocks

	Mean	Std. dev	Skew	Kurt	Max	Min	Q(15)	$Q_2(15)$
Durations								
RY	0.99	1.14	2.50	12.36	16.68	0.01	1325	387
PDG	0.93	1.50	7.68	170.59	63.33	0.01	5015	185
Returns								
RY	0.000	0.815	0.005	7.560	5.960	-6.422	8196	15417
PDG	0.001	0.890	-0.009	6.735	6.404	-6.421	2355	3118

Note: The sample period runs from April 1st to June 30, 2001. It consists of 51,660 observations for the RY stock and 27,956 observations for the PDG stock. *Mean* is the sample mean, *Std. dev* is the sample standard deviation, *Skew* is the sample skewness coefficient, *Kurt* is the sample kurtosis, *Max* is the sample maximum, *Min* is the sample minimum. $Q(15)$ is the Ljung-Box test statistics with 15 lags, $Q_2(15)$ is the Ljung-Box test statistics applied to squared returns using 15 lags. The associated 95% critical value is 24.996.

Table 3.3: Estimates of ACD-GARCH models

	RY		PDG	
Parameter	Estimate	t-Student	Estimate	t-Student
β_0	-0.017	-30.46	-0.030	-19.26
α_1	0.075	129.13	0.137	86.58
α_2	-0.057	-99.32	-0.105	-67.24
β_1	1.391	265.93	1.421	313.17
β_2	-0.408	-77.41	-0.434	-95.49
γ_1	0.425	95.04	0.391	55.36
γ_2	4.444	50.26	3.772	29.42
ϕ_1	0.069	4.11	0.078	2.58
θ_1	-0.605	-48.50	-0.369	-13.34
ω	-0.032	-9.27		
a_1	0.242	22.58	0.221	8.25
$\widetilde{\beta_1}$	0.921	232.00	0.778	19.59
γ	0.055	3.15	0.057	2.33
p-value $Q_d(15)$	0.182		0.611	
p-value $Q(15)$	0.000		0.208	
p-value $Q_2(15)$	0.002		0.833	

Note: This table contains the parameters estimates of the ACD-GARCH models for RY and PDG. *t-Student* are the t-statistics associated with the parameters estimates. *p-value* $Q_d(15)$ represents the p-value associated with the Ljung-Box test statistic computed with 15 lags applied to the ACD standardized residuals. *p-value* $Q(15)$ and *p-value* $Q_2(15)$ represent, respectively, the p-values of the Ljung-Box test statistic computed with 15 lags for serial correlation in the standardized EGARCH residuals and in their squares.

Table 3.4: IVaR results for RY

	Kupiec test				DQ test				
Level T	5%	2.5%	1%	0.5%	5%	2.5%	1%	0.5%	N
15	**0.000**	**0.000**	0.154	0.694	**0.000**	**0.021**	0.879	0.672	2470
25	**0.036**	0.231	0.928	0.317	0.365	0.344	0.360	0.003	1434
35	0.153	0.792	0.969	0.624	0.208	0.444	0.999	0.329	1012
45	0.862	0.914	0.676	0.632	0.974	0.881	0.999	0.982	781
90	0.592	0.903	0.651	0.957	0.220	0.847	0.992	0.999	385

Note: This table contains the p-values for Kupiec's test and the DQ test of Engle and Manganelli (2004) using 5 lags and the current VaR as explicative variable. T is the length of the interval used for simulation. N represents the number of intervals used for validation. Bold entries denote a failure of the model at the 95% confidence level since the p-values are inferior to 0.05.

Table 3.5: IVaR results for PDG

	Kupiec test				DQ test				
Level T	5%	2.5%	1%	0.5%	5%	2.5%	1%	0.5%	N
15	**0.000**	**0.018**	0.577	0.560	**0.009**	0.304	**0.041**	**0.000**	1294
25	0.642	0.837	0.837	0.678	0.387	0.251	0.063	**0.000**	755
35	0.762	0.944	0.274	0.675	0.136	**0.001**	0.991	0.992	530
45	0.121	0.692	0.662	0.531	**0.038**	0.670	0.568	0.711	409
90	0.762	0.215	0.994	0.381	0.544	0.590	0.934	0.645	201

Note: This table contains the p-values for Kupiec's test and the DQ test of Engle and Manganelli (2004) using 5 lags and the current VaR as explicative variable. T is the length of the interval used for simulation. N represents the number of intervals used for validation. Bold entries denote a failure of the model at the 95% confidence level since the p-values are inferior to 0.05.